

# WEAK PROREGULARITY, WEAK STABILITY, AND THE NONCOMMUTATIVE MGM EQUIVALENCE

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ABSTRACT. Let  $A$  be a commutative ring, and let  $\mathfrak{a}$  be a finitely generated ideal in it. It is known that a necessary and sufficient condition for the derived  $\mathfrak{a}$ -torsion and  $\mathfrak{a}$ -adic completion functors to be nicely behaved is the *weak proregularity* of  $\mathfrak{a}$ . In particular, the *MGM Equivalence* holds.

Because weak proregularity is defined in terms of elements of the ring (it involves limits of Koszul complexes), it is not suitable for noncommutative ring theory.

In this paper we introduce a new condition on a torsion class  $\mathcal{T}$  in a module category: *weak stability*. Our first main theorem says that in the commutative case, the ideal  $\mathfrak{a}$  is weakly proregular if and only if the corresponding torsion class  $\mathcal{T}$  is weakly stable.

We then study weak stability of torsion classes in module categories over noncommutative rings. There are three main theorems in this context:  $\triangleright$  For a torsion class  $\mathcal{T}$  that is weakly stable, quasi-compact and finite dimensional, the right derived torsion functor is isomorphic to a left derived tensor functor.  $\triangleright$  The *Noncommutative MGM Equivalence*, that holds under the same assumptions on  $\mathcal{T}$ .  $\triangleright$  A theorem about *symmetric derived torsion* for complexes of bimodules. This last theorem is a generalization of a result of Van den Bergh from 1997, and corrects an error in a paper of Yekutieli & Zhang from 2003.

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## 0. INTRODUCTION

Let  $A$  be a commutative ring, and let  $\mathfrak{a}$  be a finitely generated ideal in  $A$ . There are two functors on the category of  $A$ -modules  $M(A)$  that the ideal  $\mathfrak{a}$  determines: the  $\mathfrak{a}$ -torsion functor  $\Gamma_{\mathfrak{a}}$ , and the  $\mathfrak{a}$ -adic completion functor  $\Lambda_{\mathfrak{a}}$ . These are idempotent additive functors:  $\Gamma_{\mathfrak{a}} \circ \Gamma_{\mathfrak{a}} \cong \Gamma_{\mathfrak{a}}$  and  $\Lambda_{\mathfrak{a}} \circ \Lambda_{\mathfrak{a}} \cong \Lambda_{\mathfrak{a}}$ .

The functors  $\Gamma_{\mathfrak{a}}$  and  $\Lambda_{\mathfrak{a}}$  seem as though they could be adjoint to each other. This is false however. What is true is that under suitable assumptions, the derived functors

$$R\Gamma_{\mathfrak{a}}, L\Lambda_{\mathfrak{a}} : D(A) \rightarrow D(A)$$

are adjoints to each other. Here  $D(A)$  is the (unbounded) derived category of  $A$ -modules. The most general condition under which this is known to hold is when the ideal  $\mathfrak{a}$  is *weakly proregular*.

By definition, the ideal  $\mathfrak{a}$  is weakly proregular if it can be generated by a *weakly proregular sequence*. A sequence of elements  $\mathbf{a} = (a_1, \dots, a_n)$  in  $A$  is called weakly proregular if a rather complicated condition is satisfied by the *Koszul complexes* associated to powers of  $\mathbf{a}$ ; see Definition 4.1. This condition was first stated in [LC], but the name was only given in [AJL, Correction]. If  $A$  is noetherian, then any ideal in it is weakly proregular; but there are many non-noetherian examples (see Examples 4.9-4.11).

The next theorem is the culmination of results by Grothendieck [LC], Matlis [Ma], Greenlees and May [GM], Alonso, Jeremias and Lipman [AJL], Schenzel [Sch] and Porta, Shaul and Yekutieli [PSY1]. There is parallel recent work on these matters by Positselski [Po].

Given an ideal  $\mathfrak{a} \subseteq A$ , let  $D(A)_{\mathfrak{a}\text{-tor}}$  and  $D(A)_{\mathfrak{a}\text{-com}}$  be the essential images of the functors  $R\Gamma_{\mathfrak{a}}$  and  $L\Lambda_{\mathfrak{a}}$ .

**Theorem 0.1** (MGM Equivalence, [PSY1]). *Let  $\mathfrak{a}$  be a weakly proregular ideal in a commutative ring  $A$ .*

- (1) *The functor  $L\Lambda_{\mathfrak{a}}$  is right adjoint to the functor  $R\Gamma_{\mathfrak{a}}$ .*
- (2) *The functors  $R\Gamma_{\mathfrak{a}}$  and  $L\Lambda_{\mathfrak{a}}$  are idempotent.*
- (3) *The categories  $D(A)_{\mathfrak{a}\text{-tor}}$  and  $D(A)_{\mathfrak{a}\text{-com}}$  are full triangulated subcategories of  $D(A)$ .*
- (4) *The functor*

$$L\Lambda_{\mathfrak{a}} : D(A)_{\mathfrak{a}\text{-tor}} \rightarrow D(A)_{\mathfrak{a}\text{-com}}$$

*is an equivalence of triangulated categories, with quasi-inverse  $R\Gamma_{\mathfrak{a}}$ .*

Actually, item (1) of this theorem is usually called *GM Duality*, and it is [PSY1, Erratum, Theorem 9]. Item (2) means that  $R\Gamma_{\mathfrak{a}} \circ R\Gamma_{\mathfrak{a}} \cong R\Gamma_{\mathfrak{a}}$  and  $L\Lambda_{\mathfrak{a}} \circ L\Lambda_{\mathfrak{a}} \cong L\Lambda_{\mathfrak{a}}$ . Item (4) is properly MGM Equivalence, and it is a slight rephrasing of [PSY1, Theorem 1.1].

*The goal of this paper is to find a noncommutative analogue of weak proregularity, and to prove a suitable version of the MGM Equivalence.* It should be emphasized that all prior characterizations of weak proregularity were in terms of elements (formulas involving limits of Koszul complexes). Such formulas rarely make any sense in the noncommutative setting.

Let  $A$  be a noncommutative ring. By default  $A$ -modules are left modules. Let  $M(A)$  be the category of (left)  $A$ -modules. Recall that a *torsion class* in  $M(A)$  is a class of objects  $T \subseteq M(A)$  that is closed under taking subobjects, quotients, extensions and infinite direct sums. (In [Ste] and other texts, this is called a hereditary

torsion class.) A module  $M \in \mathbf{M}(A)$  is said to be a  $\mathbf{T}$ -torsion module if it belongs to  $\mathbf{T}$ . The torsion class  $\mathbf{T}$  gives rise to a *torsion functor*  $\Gamma_{\mathbf{T}}$ , that is a left exact additive functor from  $\mathbf{M}(A)$  to itself. For an  $A$ -module  $M$ , the module  $\Gamma_{\mathbf{T}}(M)$  is the biggest  $\mathbf{T}$ -torsion submodule of  $M$ . Thus  $M \in \mathbf{T}$  if and only if  $\Gamma_{\mathbf{T}}(M) = M$ . Following [YZ], we say that an  $A$ -module  $M$  is  $\mathbf{T}$ -flasque if  $R^q\Gamma_{\mathbf{T}}(M) = 0$  for all  $q > 0$ . Any injective  $A$ -module is  $\mathbf{T}$ -flasque, but often there are many more.

Here is the categorical notion that we propose as a generalization of weak proregularity.

**Definition 0.2.** Let  $\mathbf{T} \subseteq \mathbf{M}(A)$  be a torsion class. We call  $\mathbf{T}$  a *weakly stable torsion class* if for any injective  $A$ -module  $I$ , the module  $\Gamma_{\mathbf{T}}(I)$  is  $\mathbf{T}$ -flasque.

The name “weakly stable” reflects the standard usage of the name “stable”:  $\mathbf{T}$  is called a stable torsion class if for any injective  $A$ -module  $I$ , the module  $\Gamma_{\mathbf{T}}(I)$  is injective. See [Ste].

When  $A$  is commutative and  $\mathfrak{a}$  is a finitely generated ideal in it, the  $\mathfrak{a}$ -torsion class  $\mathbf{T} \subseteq \mathbf{M}(A)$  is the class of modules  $M$  such that  $\Gamma_{\mathfrak{a}}(M) = M$ ; and the torsion functor is  $\Gamma_{\mathbf{T}} = \Gamma_{\mathfrak{a}}$ .

Here is our first main result.

**Theorem 0.3.** *Let  $A$  be a commutative ring, let  $\mathbf{a}$  be a finite sequence of elements of  $A$ , let  $\mathfrak{a}$  be the ideal generated by  $\mathbf{a}$ , and let  $\mathbf{T}$  be the associated torsion class in  $\mathbf{M}(A)$ . The following two conditions are equivalent:*

- (i) *The sequence  $\mathbf{a}$  is weakly proregular.*
- (ii) *The torsion class  $\mathbf{T}$  is weakly stable.*

This is repeated as Theorem 4.12 in Section 4 and proved there. For other examples of weakly stable torsion classes, see Examples 3.9-3.15.

In order to state a noncommutative analogue of the MGM Equivalence, we must first introduce a suitable categorical framework. There are several ingredients involved:

- Certain properties of triangulated functors and torsion classes (namely: quasi-compact, weakly stable, finite dimensional and idempotent). This is done in Sections 1-3.
- Derived categories of bimodules, derived functors between them, and the monoidal structure on  $\mathbf{D}(A^{\text{en}})$ . This is done in Section 5.
- Idempotent copointed objects in the monoidal category  $\mathbf{D}(A^{\text{en}})$ , and the triangulated functors they induce by monoidal actions. See Section 6.

In this paper we assume that the ring  $A$  is central and flat over a commutative base ring  $\mathbb{K}$ . The opposite ring is  $A^{\text{op}}$ , and the enveloping ring is  $A^{\text{en}} := A \otimes_{\mathbb{K}} A^{\text{op}}$ . Thus, above,  $\mathbf{D}(A^{\text{en}})$  is the derived category of  $A$ -bimodules.

The flatness assumption is only for the sake of simplicity. A general treatment (not assuming flatness over  $\mathbb{K}$ ) requires the use of DG rings, and is substantially more involved. See Remarks 5.22 and 8.15.

Here is our second main result. A torsion class  $\mathbf{T}$  in  $\mathbf{M}(A)$  gives rise to a derived torsion functor  $R\Gamma_{\mathbf{T}}$  on the derived category  $\mathbf{D}(A \otimes_{\mathbb{K}} B^{\text{op}})$ , where  $B$  is any flat central  $\mathbb{K}$ -ring. There is a left monoidal action  $- \otimes_A^L -$  of  $\mathbf{D}(A^{\text{en}})$  on  $\mathbf{D}(A \otimes_{\mathbb{K}} B^{\text{op}})$ .

**Theorem 0.4** (Representability of Derived Torsion). *Let  $A$  and  $B$  be flat central  $\mathbb{K}$ -rings. Let  $\mathbf{T}$  be a quasi-compact, finite dimensional, weakly stable torsion class*

in  $M(A)$ . Define the object

$$P := R\Gamma_T(A) \in D(A^{\text{en}}).$$

Then there is an isomorphism

$$P \otimes_A^L - \cong R\Gamma_T$$

of triangulated functors from  $D(A \otimes_{\mathbb{K}} B^{\text{op}})$  to itself.

This is part of Theorem 7.12, which is more detailed.

There is a canonical morphism  $\rho : P \rightarrow A$  in  $D(A^{\text{en}})$ . In Theorem 7.18 we prove that the pair  $(P, \rho)$  is an *idempotent copointed object* in the monoidal category  $D(A^{\text{en}})$ , in the sense of Definition 6.2.

In the commutative weakly proregular situation, where  $T$  is the torsion class defined by an ideal  $\mathfrak{a} \subseteq A$  and  $\mathbb{K} = A$ , the object  $P = R\Gamma_T(A)$  in  $D(A)$  is represented by the *infinite dual Koszul complex*  $K_{\infty}^{\vee}(A; \mathfrak{a})$  associated to a weakly proregular sequence  $\mathfrak{a}$  that generates  $\mathfrak{a}$ . Positselski [Po] calls the object  $P$  a *dedualizing complex*.

Theorems 7.12, 7.18 and 6.17 combined are the technical results needed to prove our remaining main theorems, that are stated below.

Let  $P = R\Gamma_T(A) \in D(A^{\text{en}})$  be as in Theorem 0.4. It gives rise to a triangulated functor

$$(0.5) \quad G_T : D(A) \rightarrow D(A), \quad G_T := R\text{Hom}_A(P, -).$$

Define  $D(A)_{T\text{-tor}}$  and  $D(A)_{T\text{-com}}$  to be the essential images of the functors  $R\Gamma_T$  and  $G_T$  respectively.

**Theorem 0.6** (Noncommutative MGM Equivalence). *Let  $A$  be a flat central  $\mathbb{K}$ -ring, and let  $T$  be a quasi-compact, weakly stable, finite dimensional torsion class in  $M(A)$ . Then:*

- (1) *The functor  $G_T$  is right adjoint to  $R\Gamma_T$ .*
- (2) *The functors  $R\Gamma_T$  and  $G_T$  are idempotent.*
- (3) *The categories  $D(A)_{T\text{-tor}}$  and  $D(A)_{T\text{-com}}$  are full triangulated subcategories of  $D(A)$ .*
- (4) *The functor*

$$R\Gamma_T : D(A)_{T\text{-com}} \rightarrow D(A)_{T\text{-tor}}$$

*is an equivalence, with quasi-inverse  $G_T$ .*

This is repeated – in greater detail – as Theorem 8.3 in Section 8 and proved there.

Now to our fourth main result. We consider flat central  $\mathbb{K}$ -rings  $A$  and  $B$ . The category of  $\mathbb{K}$ -central  $A$ - $B$ -bimodules is  $M(A \otimes_{\mathbb{K}} B^{\text{op}})$ . Let  $T \subseteq M(A)$  and  $S^{\text{op}} \subseteq M(B^{\text{op}})$  be torsion classes. These lift, or extend, to torsion classes

$$T, S^{\text{op}} \subseteq M(A \otimes_{\mathbb{K}} B^{\text{op}}),$$

defined as follows: a bimodule  $M \in M(A \otimes_{\mathbb{K}} B^{\text{op}})$  is  $T$ -torsion if it is so after forgetting the  $B$ -module structure. Likewise (but on reversed sides) for  $S^{\text{op}}$ -torsion. There are corresponding right derived torsion functors

$$R\Gamma_T, R\Gamma_{S^{\text{op}}} : D(A \otimes_{\mathbb{K}} B^{\text{op}}) \rightarrow D(A \otimes_{\mathbb{K}} B^{\text{op}}).$$

A complex  $M \in D(A \otimes_{\mathbb{K}} B^{\text{op}})$  is said to have *symmetric derived  $T$ - $S^{\text{op}}$ -torsion* if

$$H^q(R\Gamma_T(M)) \in S^{\text{op}} \quad \text{and} \quad H^q(R\Gamma_{S^{\text{op}}}(M)) \in T$$

for all  $q$ .

**Theorem 0.7** (Symmetric Derived Torsion). *Let  $A$  and  $B$  be flat central  $\mathbb{K}$ -rings, and let  $\mathsf{T} \subseteq \mathsf{M}(A)$  and  $\mathsf{S}^{\text{op}} \subseteq \mathsf{M}(B^{\text{op}})$  be quasi-compact, weakly stable, finite dimensional torsion classes. Let  $M \in \mathsf{D}(A \otimes_{\mathbb{K}} B^{\text{op}})$  be a complex with symmetric derived  $\mathsf{T}$ - $\mathsf{S}^{\text{op}}$ -torsion. Then there is an isomorphism*

$$\mathsf{R}\Gamma_{\mathsf{T}}(M) \cong \mathsf{R}\Gamma_{\mathsf{S}^{\text{op}}}(M)$$

*in  $\mathsf{D}(A \otimes_{\mathbb{K}} B^{\text{op}})$ . Moreover, this isomorphism is functorial in such complexes  $M$ .*

Theorem 0.7 is repeated as Theorem 8.9 in Section 8 and proved there. This theorem is a correction of [YZ, Theorem 1.23]; see Remark 8.16 for details.

Theorems 0.6 and 0.7 are expected to serve as the foundation for a proof (along the lines of the proof by Van den Bergh in [VdB]) of the existence of a balanced dualizing complex over a noncommutative ring  $A$  that is noetherian, semilocal, complete and of mixed characteristics (i.e. it does not contain a field). This is outlined in the lecture notes [Ye7], and is work in progress [VY].

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## 1. QUASI-COMPACT FINITE DIMENSIONAL FUNCTORS

In this section we discuss several finiteness properties of additive functors, that shall play a role in our work.

Let  $A$  be a ring. We work with left  $A$ -modules. These notations are used: the abelian category of  $A$ -modules is  $\mathsf{M}(A)$ , the category of complexes  $A$ -modules is  $\mathsf{C}(A)$ , its homotopy category is  $\mathsf{K}(A)$ , and the derived category is  $\mathsf{D}(A)$ . The localization functor is the triangulated functor  $\mathsf{Q} : \mathsf{K}(A) \rightarrow \mathsf{D}(A)$ . As usual,  $\mathsf{D}^+(A)$ ,  $\mathsf{D}^-(A)$  and  $\mathsf{D}^b(A)$  are the full subcategories of  $\mathsf{D}(A)$  on the complexes with bounded below, bounded above and bounded cohomologies, respectively.

For the purpose of describing vanishing conditions for complexes and functors, we shall use the following numerical conventions. By *generalized integer* we mean an element of the ordered set  $\mathbb{Z} \cup \{\pm\infty\}$ . A generalized integer  $n$  will be called *finite* if  $n < \infty$ , i.e. if  $n \in \mathbb{Z} \cup \{-\infty\}$ . This somewhat unusual choice of nomenclature will be quite handy.

Given generalized integers  $d_0 \leq d_1$ , the *integer interval* they bound is

$$[d_0, d_1] := \{i \in \mathbb{Z} \mid d_0 \leq i \leq d_1\}.$$

If  $[e_0, e_1]$  is another integer interval, then we let

$$[d_0, d_1] + [e_0, e_1] := [d_0 + e_0, d_1 + e_1].$$

There is also the empty interval  $\emptyset$ , and for any interval  $S$  we let  $S + \emptyset := \emptyset$ . The integer intervals are partially ordered by inclusion.

An integer interval  $S$  has a supremum  $\sup(S)$  and an infimum  $\inf(S)$ , that are both generalized integers. The amplitude of  $S$  is

$$\text{amp}(S) := \sup(S) - \inf(S) \in \mathbb{N} \cup \{\pm\infty\}.$$

Note that for a nonempty interval  $S = [d_0, d_1]$  we have  $\sup(S) = d_1$ ,  $\inf(S) = d_0$  and  $\text{amp}(S) = d_1 - d_0 \in \mathbb{N} \cup \{\infty\}$ . For the empty interval  $S = \emptyset$  we have  $\sup(S) = -\infty$ ,  $\inf(S) = \infty$  and  $\text{amp}(S) = -\infty$ .

Let  $N = \bigoplus_{i \in \mathbb{Z}} N^i$  be a graded  $A$ -module. The *concentration* of  $N$  is the smallest integer interval  $\text{con}(N)$  containing the set  $\{i \in \mathbb{Z} \mid N^i \neq 0\}$ . We use the

abbreviations  $\inf(N) := \inf(\operatorname{con}(N))$ ,  $\sup(N) := \sup(\operatorname{con}(N))$  and  $\operatorname{amp}(N) := \operatorname{amp}(\operatorname{con}(N))$ .

Using this numerical terminology, a complex  $M$  belongs to  $\mathbf{D}^b(A)$  if and only if  $\operatorname{amp}(\mathbf{H}(M)) < \infty$ ;  $M \in \mathbf{D}^+(A)$  if and only if  $\inf(\mathbf{H}(M)) > -\infty$ ; etc.

It is well-known (see e.g. [Spa], [Ke], [SP, Chapter 09JD] or [Ye6, Subsection 10.5]) that any complex  $M \in \mathbf{C}(A)$  admits a quasi-isomorphism  $M \rightarrow I$ , where  $I$  is a K-injective complex, each  $I^q$  is an injective  $A$ -module, and  $\inf(I) = \inf(\mathbf{H}(M))$ .

Let  $B$  be another ring, and let  $F : \mathbf{M}(A) \rightarrow \mathbf{M}(B)$  be an additive functor. The functor  $F$  extends in the obvious way to a functor  $F : \mathbf{C}(A) \rightarrow \mathbf{C}(B)$  on complexes of modules, and this induces a triangulated functor  $F : \mathbf{K}(A) \rightarrow \mathbf{K}(B)$ . The triangulated functor  $F$  admits a right derived functor  $(\mathbf{R}F, \xi^{\mathbf{R}})$ . Recall that

$$\mathbf{R}F : \mathbf{D}(A) \rightarrow \mathbf{D}(B)$$

is a triangulated functor, and

$$\xi^{\mathbf{R}} : F \rightarrow \mathbf{R}F \circ Q$$

is a morphism of triangulated functors  $\mathbf{K}(A) \rightarrow \mathbf{D}(B)$  that has a certain universal property. The right derived functor  $(\mathbf{R}F, \xi^{\mathbf{R}})$  can be constructed using a K-injective presentation: for any  $M \in \mathbf{K}(A)$  we choose a K-injective resolution  $\zeta_M : M \rightarrow I_M$ , and then we take  $\mathbf{R}F(M) := F(I_M)$  and  $\xi_M^{\mathbf{R}} := F(\zeta_M)$ . See [Ye6, Subsection 8.3].

If the additive functor  $F$  is left exact, then the classical right derived functors of  $F$  are

$$\mathbf{R}^q F = \mathbf{H}^q(\mathbf{R}F) : \mathbf{M}(A) \rightarrow \mathbf{M}(B).$$

Recall that the *right cohomological dimension* of  $F$  is

$$n := \sup \{q \in \mathbb{N} \mid \mathbf{R}^q F \neq 0\} \in \mathbb{N} \cup \{\pm\infty\}.$$

If  $F \neq 0$  then  $n \in \mathbb{N} \cup \{\infty\}$ , but for  $F = 0$  the dimension is  $n = -\infty$ . Our convention regarding finiteness (a generalized integer  $n$  is finite if and only if  $n < \infty$ ) was designed to give the zero functor finite right cohomological dimension.

**Definition 1.1.** Let  $F : \mathbf{M}(A) \rightarrow \mathbf{M}(B)$  be a left exact additive functor. A module  $I \in \mathbf{M}(A)$  is called a *right  $F$ -acyclic module* if  $\mathbf{R}^q F(I) = 0$  for all  $q > 0$ .

Of course any injective  $A$ -module is a right  $F$ -acyclic module. But often there are many more right  $F$ -acyclic modules.

**Definition 1.2.** Let  $\mathbf{M}$  and  $\mathbf{N}$  be additive categories that admit infinite direct sums, and let  $F : \mathbf{M} \rightarrow \mathbf{N}$  be an additive functor. The functor  $F$  is called *quasi-compact* if it commutes with infinite direct sums. Namely, for any collection  $\{M_x\}_{x \in X}$  of objects of  $\mathbf{M}$ , indexed by some set  $X$ , the canonical morphism

$$\bigoplus_{x \in X} F(M_x) \rightarrow F\left(\bigoplus_{x \in X} M_x\right)$$

in  $\mathbf{N}$  is an isomorphism.

The name “quasi-compact functor” is inspired by the property of pushforward of quasi-coherent sheaves along a quasi-compact map of schemes.

**Lemma 1.3.** Let  $F : \mathbf{M}(A) \rightarrow \mathbf{M}(B)$  be a left exact additive functor, and assume that all the functors  $\mathbf{R}^q F$  are quasi-compact. Let  $\{I_x\}_{x \in X}$  be a collection of right  $F$ -acyclic  $A$ -modules. Then the  $A$ -module  $I := \bigoplus_{x \in X} I_x$  is right  $F$ -acyclic.

*Proof.* Take any  $q > 0$ . Because  $R^q F$  is quasi-compact, the canonical homomorphism

$$\bigoplus_{x \in X} R^q F(I_x) \rightarrow R^q F(I)$$

in  $M(B)$  is an isomorphism. But by assumption,  $R^q F(I_x) = 0$  for all  $x$ .  $\square$

**Definition 1.4.** Let  $F : M(A) \rightarrow M(B)$  be an additive functor. A complex  $I \in C(A)$  is called a *right  $F$ -acyclic complex* if the morphism  $\xi_I^R : F(I) \rightarrow RF(I)$  in  $D(B)$  is an isomorphism.

Of course any K-injective complex is right  $F$ -acyclic, but often there are many more.

In case  $F$  is a left exact functor, so that Definition 1.1 applies, it is easy to see that an  $A$ -module  $I$  is right  $F$ -acyclic if and only if it is right  $F$ -acyclic as a complex; i.e. Definitions 1.4 and 1.1 agree in this case.

**Lemma 1.5.** *Let  $F : M(A) \rightarrow M(B)$  be a left exact additive functor, and let  $I \in C(A)$  be a complex such that each of the modules  $I^q$  is right  $F$ -acyclic. If  $I$  is a bounded below complex, or if  $F$  has finite right cohomological dimension, then  $I$  is a right  $F$ -acyclic complex.*

*Proof.* This is contained in the proof of [RD, Corollary I.5.3], but for the sake of completeness we give a proof here.

We can assume that neither  $F$  nor  $I$  are zero. First let us assume that  $I$  is bounded below. We can find a quasi-isomorphism  $I \rightarrow J$ , where  $J$  is a bounded below complex of injective modules, and thus it is a K-injective complex. We must show that  $F(I) \rightarrow F(J)$  is a quasi-isomorphism. This amounts to showing that the complex  $F(K)$  is acyclic, where  $K$  is the cone on the quasi-isomorphism  $I \rightarrow J$ . Thus we reduce the problem to proving that for an acyclic bounded below complex  $I$  made up of right  $F$ -acyclic modules, the complex  $F(I)$  is acyclic.

Say  $\inf(I) = d_0 \in \mathbb{Z}$ . For any  $q$  let  $Z^q(I) := \text{Ker}(I^q \rightarrow I^{q+1})$ , the module of degree  $q$  cocycles. The acyclicity of the complex  $I$  says that we have short exact sequences

$$0 \rightarrow Z^q(I) \rightarrow I^q \rightarrow Z^{q+1}(I) \rightarrow 0.$$

By induction on  $q \geq d_0$  we prove that each  $Z^q(I)$  is a right  $F$ -acyclic module. Therefore the sequences

$$0 \rightarrow F(Z^q(I)) \rightarrow F(I^q) \rightarrow F(Z^{q+1}(I)) \rightarrow 0$$

are exact for all  $q$ . Splicing together three sequences like this, for  $q-1$ ,  $q$  and  $q+1$ , shows that the sequence

$$F(I^{q-1}) \rightarrow F(I^q) \rightarrow F(I^{q+1})$$

is exact. Thus the complex  $F(I)$  is acyclic.

Now  $I$  is no longer bounded below, but the right cohomological dimension of  $F$  is finite, say  $n \in \mathbb{N}$ . Let  $I \rightarrow J$  be a quasi-isomorphism, where  $J$  is a K-injective complex made up of injective modules. We must show that  $F(I) \rightarrow F(J)$  is a quasi-isomorphism. As above, by replacing  $I$  with the cone on  $I \rightarrow J$ , we reduce the problem to proving that for an acyclic complex  $I$  made up of right  $F$ -acyclic modules, the complex  $F(I)$  is acyclic.

Fix an integer  $q$ . Let  $M := Z^q(I)$ , and let

$$J := (\cdots \rightarrow 0 \rightarrow I^q \rightarrow I^{q+1} \rightarrow \cdots),$$

the complex with  $I^q$  placed in degree 0 (so  $J$  is a shift of a stupid truncation of  $I$ ). By the previous part of the proof we know that  $\mathrm{R}F(J) \cong F(J)$  in  $\mathrm{D}(A)$ . We have a quasi-isomorphism  $M \rightarrow J$ . Hence, for every  $p > n$ , by our assumption on  $F$  we have

$$0 = \mathrm{H}^p(\mathrm{R}F(M)) \cong \mathrm{H}^p(\mathrm{R}F(J)) \cong \mathrm{H}^p(F(J)) \cong \mathrm{H}^{p+q}(F(I)).$$

Since  $q$  was chosen arbitrarily, we conclude that  $F(I)$  is acyclic.  $\square$

Let  $F : \mathrm{D}(A) \rightarrow \mathrm{D}(B)$  be a triangulated functor, and let  $\mathbf{E} \subseteq \mathrm{D}(A)$  be a class of objects. The *cohomological displacement* of  $F$  relative to  $\mathbf{E}$  is the smallest integer interval  $S$  such that

$$\mathrm{con}(\mathrm{H}(F(M))) \subseteq \mathrm{con}(\mathrm{H}(M)) + S$$

for every  $M \in \mathbf{E}$ . The *cohomological dimension* of  $F$  relative to  $\mathbf{E}$  is the amplitude of its cohomological displacement. Note that for the zero functor  $F$ , its cohomological displacement is the empty interval, and its cohomological dimension is  $-\infty$ . It is clear that if  $\mathbf{E} \subseteq \mathbf{E}'$ , then the cohomological dimension of  $F$  relative to  $\mathbf{E}'$  is greater or equal to its dimension relative to  $\mathbf{E}$ .

**Definition 1.6.** Let  $F : \mathrm{D}(A) \rightarrow \mathrm{D}(B)$  be a triangulated functor. The *cohomological dimension* of  $F$  is its cohomological dimension relative to  $\mathrm{D}(A)$ .

Note that for a left exact additive functor  $F : \mathbf{M}(A) \rightarrow \mathbf{M}(B)$ , the cohomological dimension of the triangulated functor  $\mathrm{R}F : \mathrm{D}(A) \rightarrow \mathrm{D}(B)$  relative to the subclass  $\mathbf{M}(A) \subseteq \mathrm{D}(A)$  equals the right cohomological dimension of the functor  $F$ . As mentioned above, this number is less than or equal to the cohomological dimension of  $\mathrm{R}F$ . But it turns out that these two generalized integers are equal:

**Proposition 1.7.** *Let  $F : \mathbf{M}(A) \rightarrow \mathbf{M}(B)$  be a left exact additive functor. Then the cohomological dimension of  $\mathrm{R}F : \mathrm{D}(A) \rightarrow \mathrm{D}(B)$  equals the right cohomological dimension of  $F$ .*

*Proof.* We can assume that  $F \neq 0$ , and it has finite right cohomological dimension, say  $n \in \mathbb{N}$ . We shall prove that the cohomological displacement of  $\mathrm{R}F$  is  $[d_0, d_1] := [0, n]$ . By considering a nonzero module  $M$  that is either injective and  $F(M) \neq 0$ , for which  $\inf(\mathrm{H}(\mathrm{R}F(M))) = 0$ , or for which  $\sup(\mathrm{H}(\mathrm{R}F(M))) = n$ , we see that the cohomological displacement of  $\mathrm{R}F$  contains the integer interval  $[0, n]$ . We must prove the converse, namely that for any  $M \in \mathrm{D}(A)$  the interval  $\mathrm{con}(\mathrm{H}(\mathrm{R}F(M)))$  is contained in the interval  $\mathrm{con}(\mathrm{H}(M)) + [0, n]$ . This is done by cases, and we can assume that  $M \neq 0$ .

Case 1. If  $M$  is not in  $\mathrm{D}^-(\mathbf{M})$ , i.e.  $\mathrm{con}(\mathrm{H}(M)) = [d_0, \infty]$  for some  $d_0 \in \mathbb{Z} \cup \{-\infty\}$ , then we can take a K-injective resolution  $M \rightarrow I$  such that  $\inf(I) = d_0$ . Then  $\mathrm{H}(\mathrm{R}F(M)) = \mathrm{H}(F(I))$  is concentrated in  $[d_0, \infty]$ .

Case 2. Here  $M \in \mathrm{D}^-(\mathbf{M})$ , so that  $\mathrm{con}(\mathrm{H}(M)) = [d_0, d_1]$  for some  $d_0 \in \mathbb{Z} \cup \{-\infty\}$  and  $d_1 \in \mathbb{Z}$ . We now take a K-injective resolution  $M \rightarrow I$  such that  $\inf(I) = d_0$  and  $I$  is made up of injective  $A$ -modules. We must prove that  $\mathrm{H}^q(F(I)) = 0$  for all  $q > d_1 + n$ . This is done like in the proof of Lemma 1.5. Let  $N := Z^{d_1}(I)$ , and let

$$J := (\cdots \rightarrow 0 \rightarrow I^{d_1} \rightarrow I^{d_1+1} \rightarrow \cdots),$$

the complex with  $I^{d_1}$  placed in degree 0. So there is a quasi-isomorphism  $N \rightarrow J$ , and it is an injective resolution of  $N$ . Hence for every  $p > n$  we have

$$0 = \mathrm{H}^p(\mathrm{R}F(N)) \cong \mathrm{H}^p(F(J)) \cong \mathrm{H}^{d_1+p}(F(I)).$$



□

It is known that the category  $D(A)$  has infinite direct sums, and they are the same as the direct sums in  $C(A)$ . See [BN, Lemma 1.5]. Therefore Definition 1.2 applies to triangulated functors  $F : D(A) \rightarrow D(B)$ , so we can talk about quasi-compact triangulated functors.

**Example 1.8.** Suppose  $B = \mathbb{Z}$ , and  $F : D(A) \rightarrow D(\mathbb{Z})$  is the triangulated functor  $F = R\mathrm{Hom}_A(P, -)$  for some complex  $P \in D(A)$ . Recall that the complex  $P$  is called a *compact object* of  $D(A)$  if the functor  $F$  is quasi-compact, in the sense of Definition 1.2. It is known (see [BV]) that  $P$  is a compact object of  $D(A)$  if and only if  $P$  is isomorphic to a bounded complex of finitely generated projective  $A$ -modules. Therefore in this case, if  $F$  is a quasi-compact functor then it is also a finite dimensional functor, in the sense of Definition 1.6. (In general these attributes are independent of each other.)

**Example 1.9.** Assume  $F = P \otimes_A^L -$  for some complex  $P \in D(A^{\mathrm{op}})$ ; here  $A^{\mathrm{op}}$  is the opposite ring. Again the target of  $F$  is  $D(\mathbb{Z})$ . Then  $F$  is a quasi-compact functor. It is a finite dimensional functor if and only if the complex  $P$  has finite flat dimension, i.e. if  $P$  is isomorphic to a bounded complex of flat  $A^{\mathrm{op}}$ -modules.

**Lemma 1.10.** *Let  $F : M(A) \rightarrow M(B)$  be a left exact additive functor, such that all the functors  $R^q F$  are quasi-compact. Let  $\{I_x\}_{x \in X}$  be a collection of right  $F$ -acyclic complexes in  $C(A)$ , and define  $I := \bigoplus_{x \in X} I_x$ . If  $I$  is a bounded below complex, or if  $F$  has finite right cohomological dimension, then  $I$  is a right  $F$ -acyclic complex.*

*Proof.* For every index  $x$  let choose a quasi-isomorphism  $\phi_x : I_x \rightarrow J_x$ , where  $J_x$  is a K-injective complex consisting of injective  $A$ -modules, and  $\inf(J_x) \geq \inf(I_x)$ . Define  $J := \bigoplus_{x \in X} J_x$ . We get a quasi-isomorphism

$$\phi := \bigoplus_{x \in X} \phi_x : I \rightarrow J.$$

By construction, if  $I$  is a bounded below complex, then so is  $J$ .

For every  $x$  there is a commutative diagram

$$\begin{array}{ccc} F(I_x) & \xrightarrow{F(\phi_x)} & F(J_x) \\ \xi_{I_x}^R \downarrow & & \downarrow \xi_{J_x}^R \\ RF(I_x) & \xrightarrow{RF(\phi_x)} & RF(J_x) \end{array}$$

in  $D(A)$ . The vertical arrows are isomorphisms because both  $I_x$  and  $J_x$  are right  $F$ -acyclic complexes. The morphism  $RF(\phi_x)$  is also an isomorphism. It follows that  $F(\phi_x)$  is an isomorphism in  $D(A)$ ; and therefore it is a quasi-isomorphism in  $C(A)$ .

Next consider this commutative diagram in  $C(A)$  :

$$\begin{array}{ccc} \bigoplus_{x \in X} F(I_x) & \xrightarrow{\bigoplus_{x \in X} F(\phi_x)} & \bigoplus_{x \in X} F(J_x) \\ \downarrow & & \downarrow \\ F(I) & \xrightarrow{F(\phi)} & F(J) \end{array}$$

The previous paragraph, and the fact that cohomology commutes with infinite direct sums, tell us that the top horizontal arrow is a quasi-isomorphism. Because the functor  $F = R^0F$  is quasi-compact, the vertical arrows are isomorphisms. We conclude that  $F(\phi)$  is a quasi-isomorphism.

Finally we look at this commutative diagram in  $D(A)$  :

$$\begin{array}{ccc} F(I) & \xrightarrow{F(\phi)} & F(J) \\ \xi_I^R \downarrow & & \downarrow \xi_J^R \\ RF(I) & \xrightarrow{RF(\phi)} & RF(J) \end{array}$$

We know that the morphisms  $F(\phi)$  and  $RF(\phi)$  are isomorphisms. For every  $p$ , the module  $J^p = \bigoplus_{x \in X} J_x^p$  is a direct sum of injective modules. So according to Lemma 1.3,  $J^p$  is a right  $F$ -acyclic module. By Lemma 1.5 (in either of the two cases) the complex  $J$  is a right  $F$ -acyclic complex. This says that the morphism  $\xi_J^R$  is an isomorphism. We conclude that the morphism  $\xi_I^R$  is an isomorphism too, and this says that  $I$  is a right  $F$ -acyclic complex.  $\square$

**Theorem 1.11.** *Let  $F : M(A) \rightarrow M(B)$  be a left exact additive functor. Assume that  $F$  has finite right cohomological dimension, and each  $R^qF$  is quasi-compact. Then the triangulated functor  $RF : D(A) \rightarrow D(B)$  is quasi-compact and finite dimensional.*

*Proof.* By Proposition 1.7 the functor  $RF$  is finite dimensional. We need to prove that  $RF$  commutes with infinite direct sums. Namely, consider a collection  $\{M_x\}_{x \in X}$  of complexes, and let  $M := \bigoplus_{x \in X} M_x$ . We have to prove that the canonical morphism

$$\bigoplus_{x \in X} RF(M_x) \rightarrow RF(M)$$

in  $D(A)$  is an isomorphism.

For every index  $x$  we choose a quasi-isomorphism  $\phi_x : M_x \rightarrow I_x$  in  $C(A)$ , where  $I_x$  is a K-injective complex. Define  $I := \bigoplus_{x \in X} I_x$ , so there is a quasi-isomorphism  $\phi : M \rightarrow I$  in  $C(A)$ . We get a commutative diagram

$$\begin{array}{ccc} \bigoplus_{x \in X} RF(M_x) & \xrightarrow{\bigoplus RF(\phi_x)} & \bigoplus_{x \in X} RF(I_x) \\ \downarrow & & \downarrow \\ RF(M) & \xrightarrow{RF(\phi)} & RF(I) \end{array}$$

in  $D(A)$ , in which the horizontal arrows are isomorphisms. Therefore it suffices to prove that the canonical morphism

$$(1.12) \quad \bigoplus_{x \in X} RF(I_x) \rightarrow RF(I)$$

in  $D(A)$  is an isomorphism.

Consider this commutative diagram in  $D(A)$  :

$$\begin{array}{ccc} \bigoplus_{x \in X} F(I_x) & \xrightarrow{\bigoplus \xi_{I_x}^R} & \bigoplus_{x \in X} RF(I_x) \\ \downarrow & & \downarrow \\ F(I) & \xrightarrow{\xi_I^R} & RF(I) \end{array}$$

For each  $x$  the morphism  $\xi_{I_x}^R$  is an isomorphism; and hence the top horizontal arrow is an isomorphism. Because  $F = R^0F$  is quasi-compact, the left vertical arrow is an isomorphism (in  $C(A)$ , and so also in  $D(A)$ ). By Lemma 1.10 the complex  $I$  is right  $F$ -acyclic, and therefore  $\xi_I^R$  is an isomorphism. Hence the remaining arrow is an isomorphism; but this is the morphism (1.12).  $\square$

## 2. WEAKLY STABLE AND IDEMPOTENT COPOINTED FUNCTORS

Again  $A$  is a ring. The category of left  $A$ -modules is  $M(A)$ . In this section we introduce a new property of additive functors from  $M(A)$  to itself.

**Definition 2.1.** Let  $F : M(A) \rightarrow M(A)$  be a left exact additive functor.

- (1) The functor  $F$  is called *weakly stable* if for any injective  $A$ -module  $I$ , the  $A$ -module  $F(I)$  is right  $F$ -acyclic.
- (2) The functor  $F$  is called *stable* if for any injective  $A$ -module  $I$ , the  $A$ -module  $F(I)$  is injective.

When we say that a functor  $F$  is stable or weakly stable, it is always implied that  $F$  is a left exact additive functor. Clearly stable implies weakly stable. The reason for these names will become apparent in the next section, when we talk about torsion functors.

**Definition 2.2.** Let  $N$  be an additive category (e.g.  $M(A)$  or  $D(A)$ ), with identity functor  $\text{Id}_N$ .

- (1) A *copointed additive functor* on  $N$  is a pair  $(F, \sigma)$ , consisting of an additive functor  $F : N \rightarrow N$ , and morphism of functors  $\sigma : F \rightarrow \text{Id}_N$ .
- (2) The copointed additive functor  $(F, \sigma)$  is called *idempotent* if the morphisms

$$\sigma_{F(N)}, F(\sigma_N) : F(F(N)) \rightarrow F(N)$$

are isomorphisms for all objects  $N \in N$ .

- (3) If  $N$  is a triangulated category,  $F$  is a triangulated functor, and  $\sigma$  is a morphism of triangulated functors, then we call  $(F, \sigma)$  a *copointed triangulated functor*.

The name “copointed” is explained in Remark 6.3.

Recall that the right derived functor of an additive functor  $F : M(A) \rightarrow M(A)$  is a pair  $(RF, \xi^R)$ , where  $RF : D(A) \rightarrow D(A)$  is a triangulated functor, and  $\xi^R : F \rightarrow RF \circ Q$  is a morphism of triangulated functors  $K(A) \rightarrow D(A)$ . See Section 1 for more details.

Weak stability and idempotence together have the following effect.

**Proposition 2.3.** *Let  $(F, \sigma)$  be an idempotent copointed additive functor on  $M(A)$ , and assume that  $F$  is weakly stable. If  $I$  is a right  $F$ -acyclic module, then  $F(I)$  is also a right  $F$ -acyclic module.*

*Proof.* Choose an injective resolution  $\eta : I \rightarrow J$ ; i.e.  $J$  is a complex of injectives concentrated in nonnegative degrees, and  $\eta$  is a quasi-isomorphism. Since  $H^q(F(J)) \cong R^q F(I)$ , and since  $I$  is a right  $F$ -acyclic module, we see that the homomorphism of complexes  $F(\eta) : F(I) \rightarrow F(J)$  is a quasi-isomorphism. Therefore both  $F(\eta)$  and  $RF(F(\eta))$  are isomorphisms in  $D(A)$ . The weak stability of  $F$  implies that  $F(J)$  is a bounded below complexes of right  $F$ -acyclic modules. According to Lemma 1.5 the complex  $F(J)$  is right  $F$ -acyclic. This means that the morphism  $\xi_{F(J)}^R$  is an isomorphism. The idempotence of  $F$  says that the morphisms  $\sigma_{F(I)}$  and  $\sigma_{F(J)}$  are isomorphisms (already in  $C(A)$ ). We get a commutative diagram in  $D(A)$ :

$$\begin{array}{ccccc} F(I) & \xleftarrow[\cong]{\sigma_{F(I)}} & F(F(I)) & \xrightarrow{\xi_{F(I)}^R} & RF(F(I)) \\ F(\eta) \downarrow \cong & & F(F(\eta)) \downarrow & & RF(F(\eta)) \downarrow \cong \\ F(J) & \xleftarrow[\cong]{\sigma_{F(J)}} & F(F(J)) & \xrightarrow[\cong]{\xi_{F(J)}^R} & RF(F(J)) \end{array}$$

We conclude that  $\xi_{F(I)}^R$  is an isomorphism; and this means that  $F(I)$  is right  $F$ -acyclic.  $\square$

The next lemma is a generalization of [PSY1, Proposition 3.10].

**Lemma 2.4.** *Suppose we are given a copointed additive functor  $(F, \sigma)$  on  $M(A)$ . Then there is a unique morphism*

$$\sigma^R : RF \rightarrow \text{Id}_{D(A)}$$

*of triangulated functors from  $D(A)$  to itself, satisfying this condition: for any  $M \in D(A)$  there is equality  $\sigma_M^R \circ \xi_M^R = \sigma_M$  of morphisms  $F(M) \rightarrow M$  in  $D(A)$ .*

In a commutative diagram:

$$(2.5) \quad \begin{array}{ccc} F(M) & \xrightarrow{\xi_M^R} & RF(M) \\ & \searrow \sigma_M & \downarrow \sigma_M^R \\ & & M \end{array}$$

*Proof.* The existence of the morphism  $\sigma^R$  comes for free from the universal property of the right derived functor. Still, for later reference, we give the construction.

For a K-injective complex  $I$  the morphism  $\xi_I^R : F(I) \rightarrow RF(I)$  in  $D(A)$  is an isomorphism, and we define  $\sigma_I^R : RF(I) \rightarrow I$  to be  $\sigma_I^R := \sigma_I \circ (\xi_I^R)^{-1}$ . For an arbitrary complex  $M$  we choose a quasi-isomorphism  $\eta : M \rightarrow I$  into a K-injective complex, and then we let

$$\sigma_M^R := \eta^{-1} \circ \sigma_I^R \circ RF(\eta)$$

in  $D(A)$ . The corresponding commutative diagram in  $D(A)$  is this:

$$\begin{array}{ccccc}
 & & \sigma_M & & \\
 & \curvearrowright & & \curvearrowleft & \\
 F(M) & \xrightarrow{\xi_M^R} & RF(M) & \xrightarrow{\sigma_M^R} & M \\
 \downarrow F(\eta) & & \downarrow RF(\eta) \cong & & \downarrow \eta \cong \\
 F(I) & \xrightarrow[\cong]{\xi_I^R} & RF(I) & \xrightarrow{\sigma_I^R} & I \\
 & \curvearrowleft & & \curvearrowright & \\
 & & \sigma_I & & 
 \end{array}$$

It is easy to see that the collection of morphisms  $\{\sigma_M^R\}_{M \in D(A)}$  has the desired properties.  $\square$

In this way we obtain a copointed triangulated functor  $(RF, \sigma^R)$  on  $D(A)$ .

**Theorem 2.6.** *Let  $(F, \sigma)$  be an idempotent copointed additive functor on  $M(A)$ . Assume that  $F$  is weakly stable and has finite right cohomological dimension. Then the copointed triangulated functor  $(RF, \sigma^R)$  on  $D(A)$  is idempotent.*

*Proof.* Let  $M$  be any complex of  $A$ -modules. Choose an isomorphism  $M \xrightarrow{\cong} I$  in  $D(A)$ , where  $I$  is a complex consisting of injective  $A$ -modules (e.g. take a K-injective resolution  $M \rightarrow I$ , such that  $I$  consists of injective modules). It suffices to prove that

$$\sigma_{RF(I)}^R, RF(\sigma_I^R) : RF(RF(I)) \rightarrow RF(I)$$

are isomorphisms in  $D(A)$ .

Since each  $I^q$  is an injective module, it is right  $F$ -acyclic. Since  $F$  is a weakly stable functor, each of the modules  $F(I^q)$  is right  $F$ -acyclic too. The functor  $F$  has finite right cohomological dimension, so according to Lemma 1.5, the complexes  $I$  and  $F(I)$  are both right  $F$ -acyclic complexes.

Consider the diagram

$$\begin{array}{ccccc}
 (2.7) & F(F(I)) & \xrightarrow{\xi_{F(I)}^R} & RF(F(I)) & \xrightarrow{RF(\xi_I^R)} & RF(RF(I)) \\
 & \downarrow F(\sigma_I) & & \downarrow RF(\sigma_I) & & \downarrow RF(\sigma_I^R) \\
 & F(I) & \xrightarrow{\xi_I^R} & RF(I) & \xrightarrow{\text{id}} & RF(I)
 \end{array}$$

in  $D(A)$ . The left square is commutative: it is gotten from the vertical morphism  $\sigma_I : F(I) \rightarrow I$ , to which we apply in the horizontal direction the morphism of functors  $\xi^R : F \rightarrow RF$ . The right square is also commutative: it comes from applying the functor  $RF$  to the commutative diagram

$$\begin{array}{ccc}
 F(I) & \xrightarrow{\xi_I^R} & RF(I) \\
 \sigma_I \downarrow & & \downarrow \sigma_I^R \\
 I & \xrightarrow{\text{id}} & I
 \end{array}$$

that characterizes  $\sigma_I^R$ . Because  $I$  and  $F(I)$  are right  $F$ -acyclic complexes, the morphisms  $\xi_I^R$  and  $\xi_{F(I)}^R$  are isomorphisms. Hence  $RF(\xi_I^R)$  is an isomorphism. So

the horizontal morphisms in the diagram 2.7 are all isomorphisms. We are given that  $F$  is idempotent, and thus  $F(\sigma_I)$  is an isomorphism. The conclusion of this discussion is that  $RF(\sigma_I^R)$  is an isomorphism.

Next, let  $\phi : RF(I) \rightarrow J$  be an isomorphism in  $D(A)$  to a K-injective complex  $J$ . We know that  $\xi_I^R : F(I) \rightarrow RF(I)$  is an isomorphism, and by composing them we get the isomorphism  $\phi \circ \xi_I^R : F(I) \rightarrow J$  in  $D(A)$ . Let  $\psi : F(I) \rightarrow J$  be a quasi-isomorphism in  $C(A)$  representing  $\phi \circ \xi_I^R$ . Since these are both right  $F$ -acyclic complexes, it follows that  $F(\psi) : F(F(I)) \rightarrow F(J)$  is a quasi-isomorphism. isomorphisms. Consider the following commutative diagram

$$\begin{array}{ccccccc}
 F(F(I)) & \xrightarrow{F(\psi)} & F(J) & \xrightarrow{\xi_J^R} & RF(J) & \xleftarrow{RF(\phi)} & RF(RF(I)) \\
 \downarrow \sigma_{F(I)} & & \downarrow \sigma_J & & \downarrow \sigma_J^R & & \downarrow \sigma_{RF(I)}^R \\
 F(I) & \xrightarrow{\psi} & J & \xrightarrow{\text{id}} & J & \xleftarrow{\phi} & RF(I)
 \end{array}$$

in  $D(A)$ . All horizontal arrows here are isomorphisms. We are given that  $F$  is idempotent, and thus  $\sigma_{F(I)}$  is an isomorphism. The conclusion is that  $\sigma_{RF(I)}^R$  is an isomorphism.  $\square$

Later on we shall need a notion dual to “copointed functor”.

**Definition 2.8.** Let  $\mathbf{N}$  be an additive category (e.g.  $M(A)$  or  $D(A)$ ), with identity functor  $\text{Id}_{\mathbf{N}}$ .

- (1) A *pointed additive functor* on  $\mathbf{N}$  is a pair  $(G, \tau)$ , consisting of an additive functor  $G : \mathbf{N} \rightarrow \mathbf{N}$ , and morphism of functors  $\tau : \text{Id}_{\mathbf{N}} \rightarrow G$ .
- (2) The pointed additive functor  $(G, \tau)$  is called *idempotent* if the morphisms

$$\tau_{G(N)}, G(\tau_N) : G(N) \rightarrow G(G(N))$$

are isomorphisms for all objects  $N \in \mathbf{N}$ .

- (3) If  $\mathbf{N}$  is a triangulated category,  $G$  is a triangulated functor, and  $\tau$  is a morphism of triangulated functors, then we call  $(G, \tau)$  a *pointed triangulated functor*.

**Remark 2.9.** Idempotent copointed functors already appeared in the literature under another name: idempotent comonads. Another name for (nearly) the same notion is a (Bousfield) colocalization functor, see e.g. [Kr]. Dually, idempotent pointed functors are the same thing as idempotent monads. See [nLab] for a discussion of these concepts.

In [KS, Section 4.1], what we call an idempotent pointed functor is called a projector. It is proved there that for an idempotent pointed functor  $(G, \tau)$ , and any object  $N$ , the morphisms  $\tau_{G(N)}$  and  $G(\tau_N)$  are equal. The same proof (with arrows reversed) shows that for an idempotent copointed functor  $(F, \sigma)$ , and any object  $N$ , the morphisms  $\sigma_{F(N)}$  and  $F(\sigma_N)$  are equal. We shall not require these facts.

### 3. TORSION CLASSES

Again  $A$  is a ring. The category of left  $A$ -modules is  $M(A)$ .

**Definition 3.1.** A *torsion class* in  $M(A)$  is a class of objects  $\mathbf{T} \subseteq M(A)$  that is closed under taking submodules, quotients, extensions and infinite direct sums.

**Remark 3.2.** Many texts (including [Ste, Chapter VI]) use the name “hereditary torsion class”, and the extra adjective “hereditary” indicates that  $\mathsf{T}$  is closed under taking submodules. Since this distinction never shows up in our work (all our torsion classes are hereditary), and since we have plenty of other attributes to attach to a torsion class, we decided to allow ourselves to simplify the naming.

Note that the full subcategory on a torsion class  $\mathsf{T}$  is called a *localizing subcategory* of  $\mathsf{M}(A)$ .

A torsion class  $\mathsf{T} \subseteq \mathsf{M}(A)$  induces a left exact additive functor  $\Gamma_{\mathsf{T}}$  from  $\mathsf{M}(A)$  to itself, called the *torsion functor*. The formula is this: for a module  $M$ ,  $\Gamma_{\mathsf{T}}(M)$  is the largest submodule of  $M$  that belongs to  $\mathsf{T}$ . For this reason, a module  $M$  that belongs to  $\mathsf{T}$  is called a  *$\mathsf{T}$ -torsion module*. As  $M$  varies, the inclusions  $\sigma_M : \Gamma_{\mathsf{T}}(M) \rightarrow M$  become a morphism of functors  $\sigma : \Gamma_{\mathsf{T}} \rightarrow \text{Id}_{\mathsf{M}(A)}$ , and thus the pair  $(\Gamma_{\mathsf{T}}, \sigma)$  is a copointed additive functor. This is in fact an idempotent copointed functor, since  $\Gamma_{\mathsf{T}}(\Gamma_{\mathsf{T}}(M)) = \Gamma_{\mathsf{T}}(M)$  as submodules of  $M$ . The class  $\mathsf{T}$  can be recovered from the copointed additive functor  $(\Gamma_{\mathsf{T}}, \sigma)$ , as follows:  $M \in \mathsf{T}$  if and only if  $\sigma_M : \Gamma_{\mathsf{T}}(M) \rightarrow M$  is an isomorphism.

A torsion class  $\mathsf{T}$  determines a set of left ideals  $\text{Filt}(\mathsf{T})$ , called the *Gabriel filter* of  $\mathsf{T}$ . By definition, a left ideal  $\mathfrak{a} \subseteq A$  belongs to  $\text{Filt}(\mathsf{T})$  if the left module  $A/\mathfrak{a}$  belongs to  $\mathsf{T}$ . The functor  $\Gamma_{\mathsf{T}}$  can be recovered from the Gabriel filter, as follows. We view  $\text{Filt}(\mathsf{T})$  as a partially ordered set by inclusion. Then for any module  $M$  there is equality

$$\Gamma_{\mathsf{T}}(M) = \lim_{\substack{\longrightarrow \\ \mathfrak{a} \in \text{Filt}(\mathsf{T})}} \text{Hom}_A(A/\mathfrak{a}, M)$$

of submodules of  $M$ . In other words, an element  $m \in M$  lies inside  $\Gamma_{\mathsf{T}}(M)$  if and only if  $m$  is annihilated by some left ideal  $\mathfrak{a} \in \text{Filt}(\mathsf{T})$ .

Here is a source of torsion classes.

**Definition 3.3.** Let  $A$  be a ring, and let  $\mathfrak{a}$  be a two-sided ideal in  $A$  that is finitely generated as a left ideal. For an  $A$ -module  $M$ , we define its  *$\mathfrak{a}$ -torsion submodule* to be

$$\Gamma_{\mathfrak{a}}(M) := \lim_{i \rightarrow} \text{Hom}_A(A/\mathfrak{a}^i, M) \subseteq M.$$

The class of objects  $\mathsf{T}_{\mathfrak{a}} \subseteq \mathsf{M}(A)$  is defined to be

$$\mathsf{T}_{\mathfrak{a}} := \{M \mid \Gamma_{\mathfrak{a}}(M) = M\}.$$

Note that an element  $m \in M$  belongs to  $\Gamma_{\mathfrak{a}}(M)$  if and only if  $\mathfrak{a}^i \cdot m = 0$  for  $i \gg 0$ . We require  $\mathfrak{a}$  to be finitely generated as a left ideal to ensure that  $\mathsf{T}_{\mathfrak{a}}$  is closed under extensions.

**Definition 3.4.** Let  $\mathsf{T}$  be a torsion class in  $\mathsf{M}(A)$ .

- (1) We call  $\mathsf{T}$  a *weakly stable* torsion class if the functor  $\Gamma_{\mathsf{T}}$  is weakly stable, as in Definition 2.1(1).
- (2) We call  $\mathsf{T}$  a *stable* torsion class if the functor  $\Gamma_{\mathsf{T}}$  is stable, as in Definition 2.1(2).
- (3) We call  $\mathsf{T}$  a *quasi-compact* torsion class if for every  $q \geq 0$  the functor  $R^q \Gamma_{\mathsf{T}}$  is quasi-compact, as in Definition 1.2.
- (4) The *dimension* of  $\mathsf{T}$  is defined to be the right cohomological dimension of the functor  $\Gamma_{\mathsf{T}}$ .

Item (2) in the definition above, i.e. the notion of stable torsion class, is standard – see [Ste, Section VI.7]. The rest of the definition is new.

Example 3.11 presents a torsion class that is weakly stable but not stable.

**Definition 3.5.** Let  $\mathsf{T}$  be a torsion class in  $\mathsf{M}(A)$ .

- (1) A module  $I \in \mathsf{M}(A)$  is called a  $\mathsf{T}$ -flasque module if it is right  $\Gamma_{\mathsf{T}}$ -acyclic, as in Definition 1.1.
- (2) A complex  $I \in \mathsf{C}(A)$  is called a  $\mathsf{T}$ -flasque complex if it is right  $\Gamma_{\mathsf{T}}$ -acyclic, as in Definition 1.4.

Item (1) in the definition above is copied from [YZ]. Note that for a module  $I$ , considered also as a complex, the definition is consistent.

**Proposition 3.6.** *If  $\mathsf{T}$  is a weakly stable torsion class in  $\mathsf{M}(A)$ , then any module  $M \in \mathsf{T}$  is  $\mathsf{T}$ -flasque.*

*Proof.* Take any  $M \in \mathsf{T}$ . We can embed  $M$  in an injective module  $I^0$ . But then  $M \subseteq \Gamma_{\mathsf{T}}(I^0)$ , so we have an exact sequence  $0 \rightarrow M \xrightarrow{\eta} \Gamma_{\mathsf{T}}(I^0)$ . The cokernel of  $\eta$  is in  $\mathsf{T}$ , so the process can be continued, to give an exact sequence

$$0 \rightarrow M \xrightarrow{\eta} \Gamma_{\mathsf{T}}(I^0) \rightarrow \Gamma_{\mathsf{T}}(I^1) \rightarrow \cdots,$$

where the modules  $I^q$  are injective. Writing  $J^q := \Gamma_{\mathsf{T}}(I^q)$ , we get a quasi-isomorphism of complexes  $\eta : M \rightarrow J$ .

Because  $\mathsf{T}$  is weakly stable, each  $J^q = \Gamma_{\mathsf{T}}(I^q)$  is a right  $\Gamma_{\mathsf{T}}$ -acyclic module (i.e. a  $\mathsf{T}$ -flasque module). According to Lemma 1.5,  $J$  is a right  $\Gamma_{\mathsf{T}}$ -acyclic complex (i.e. a  $\mathsf{T}$ -flasque complex). This accounts for the second isomorphism in:

$$\mathrm{H}^q(\mathrm{R}\Gamma_{\mathsf{T}}(M)) \cong \mathrm{H}^q(\mathrm{R}\Gamma_{\mathsf{T}}(J)) \cong \mathrm{H}^q(\Gamma_{\mathsf{T}}(J)) \cong \mathrm{H}^q(J) \cong \mathrm{H}^q(M).$$

The first and fourth isomorphisms come from the quasi-isomorphism  $\eta : M \rightarrow J$ . And  $\Gamma_{\mathsf{T}}(J) = J$  because  $\Gamma_{\mathsf{T}}$  is idempotent.

The conclusion is that  $\mathrm{H}^q(\mathrm{R}\Gamma_{\mathsf{T}}(M)) = 0$  for  $q > 0$ .  $\square$

As in Section 2, the copointed additive functor  $(\Gamma_{\mathsf{T}}, \sigma)$  on  $\mathsf{M}(A)$  gives rise to a copointed triangulated functor  $(\mathrm{R}\Gamma_{\mathsf{T}}, \sigma^{\mathrm{R}})$  on  $\mathsf{D}(A)$ .

**Proposition 3.7.** *Let  $\mathsf{T}$  be a weakly stable finite dimensional torsion class in  $\mathsf{M}(A)$ . The following conditions are equivalent for a complex  $M \in \mathsf{D}(A)$ .*

- (i)  $\sigma_M^{\mathrm{R}} : \mathrm{R}\Gamma_{\mathsf{T}}(M) \rightarrow M$  is an isomorphism in  $\mathsf{D}(A)$ .
- (ii)  $M \cong \mathrm{R}\Gamma_{\mathsf{T}}(N)$  in  $\mathsf{D}(A)$  for some complex  $N$ .
- (iii)  $\mathrm{H}^q(M) \in \mathsf{T}$  for all  $q$ .

*Proof.* The implications (i)  $\Rightarrow$  (ii) and (ii)  $\Rightarrow$  (iii) are obvious. Let us prove that (iii)  $\Rightarrow$  (i). Recall that  $\sigma_M^{\mathrm{R}} \circ \xi_M^{\mathrm{R}} = \sigma_M$ . First suppose  $M$  is a module in  $\mathsf{T}$ , so that  $\sigma_M$  is an isomorphism. By Proposition 3.6 the morphism  $\xi_M^{\mathrm{R}}$  is also an isomorphism. Hence  $\sigma_M^{\mathrm{R}}$  is an isomorphism.

Now to the general case:  $M$  is a complex of modules satisfying  $\mathrm{H}^q(M) \in \mathsf{T}$  for all  $q$ . According to Proposition 1.7 the functor  $\mathrm{R}\Gamma_{\mathsf{T}}$  is finite dimensional. The way-out argument of [RD, Chapter I.7], or [Ye1, Theorem 2.10], imply that  $\sigma_M^{\mathrm{R}}$  is an isomorphism.  $\square$

**Theorem 3.8.** *Let  $A$  be a ring, and let  $\mathsf{T}$  be a weakly stable finite dimensional quasi-compact torsion class in  $\mathsf{M}(A)$ . Then the copointed triangulated functor*



$(R\Gamma_{\mathbf{T}}, \sigma^R)$  on  $D(A)$  is idempotent, and the triangulated functor  $R\Gamma_{\mathbf{T}}$  is finite dimensional and quasi-compact.

*Proof.* The functor  $\Gamma_{\mathbf{T}}$  is always left exact, and the copointed additive functor  $(\Gamma_{\mathbf{T}}, \sigma)$  is always idempotent. Looking at Definition 3.4, we see that moreover the functor  $\Gamma_{\mathbf{T}}$  is weakly stable and has finite right cohomological dimension, and the functors  $R^q\Gamma_{\mathbf{T}}$  are quasi-compact. So this theorem is a special case of Theorems 1.11 and 2.6.  $\square$

Here are several examples of weakly stable torsion classes.

**Example 3.9.** Let  $A$  be a noetherian commutative ring, and let  $\mathfrak{a}$  be any ideal in  $A$ . Then  $\mathbf{T}_{\mathfrak{a}}$  is a stable torsion class in  $M(A)$ . One way to prove this is by the structure theory of injective  $A$ -modules (see the proof of [PSY1, Theorem 4.34]). By Proposition 3.6, any module  $M \in \mathbf{T}_{\mathfrak{a}}$  is  $\mathbf{T}_{\mathfrak{a}}$ -flasque.

**Example 3.10.** Let  $A$  be a commutative ring, and let  $\mathfrak{a}$  be an ideal in  $A$ . In Section 4 we will show that the ideal  $\mathfrak{a}$  is *weakly proregular* if and only if the torsion class  $\mathbf{T}_{\mathfrak{a}}$  is weakly stable.

**Example 3.11.** Here we present a torsion class that is *weakly stable but not stable*. Let  $A$  be a commutative ring that is not noetherian, let  $B := A[t]$ , the polynomial ring in one variable, and let  $\mathfrak{b} := (t) \subseteq B$ . The torsion class  $\mathbf{T} := \mathbf{T}_{\mathfrak{b}} \subseteq M(B)$  is weakly stable, by Example 4.9 and Theorem 4.12.

Now let us suppose, for the sake of contradiction, that  $\mathbf{T}$  is stable. Consider any countable collection  $\{I_p\}_{p \in \mathbb{N}}$  of injective  $A$ -modules. Then  $I := \prod_p I_p$  is injective over  $A$ , so  $\mathrm{Hom}_A(B, I)$  is injective over  $B$ . Stability of  $\mathbf{T}$  says that  $J := \Gamma_{\mathfrak{b}}(\mathrm{Hom}_A(B, I))$  is also injective over  $B$ . Since  $A \rightarrow B$  is flat, it follows that  $J$  is injective over  $A$ . But as  $A$ -modules there is an isomorphism  $B \cong A^{\oplus \mathbb{N}}$ , and thus  $\mathrm{Hom}_A(B, I) \cong I^{\times \mathbb{N}}$  and  $J \cong I^{\oplus \mathbb{N}}$ . Each  $I_p$  is a direct summand of  $I$ , and therefore  $\bigoplus_{p \in \mathbb{N}} I_p$  is a direct summand of  $J$ . The conclusion is that any countable direct sum of injective  $A$ -modules is injective. By [La, Theorem 3.46(3)] this implies that  $A$  is noetherian, contradicting our assumptions.

**Example 3.12.** Let  $A$  be a ring, and let  $\mathfrak{a}$  be a two-sided ideal in  $A$  that is finitely generated as a left ideal. In Definition 3.3 we introduced the torsion class  $\mathbf{T}_{\mathfrak{a}}$ . It is possible to show that  $\mathbf{T}_{\mathfrak{a}}$  is a stable torsion class if and only if the ideal  $\mathfrak{a}$  has the *left Artin-Rees property*, in the sense of [GW, Chapter 13].

If  $A$  happens to be a noetherian commutative ring, as in Example 3.9, then the original Artin-Rees property holds. This furnishes another proof for that case.

**Example 3.13.** Let  $A$  be a ring and let  $S$  be a left denominator set in it. Then the left Ore localization  $A_S = A[S^{-1}]$  exists. Define

$$\mathbf{T}_S := \{M \in M(A) \mid A_S \otimes_A M = 0\}.$$

This is a torsion class in  $M(A)$ . It can be shown that the torsion class  $\mathbf{T}_S$  is weakly stable if and only if it has dimension  $\leq 1$ . This will appear in the paper [Vs2].

In the special case where  $A$  is a commutative integral domain, it can be shown that this condition holds, and thus  $\mathbf{T}_S$  is weakly stable in this case.

**Example 3.14.** Let  $A$  be a ring, and let  $\mathbf{a} = (a_1, \dots, a_n)$  be a sequence of elements in it. Let  $\mathfrak{a} \subseteq A$  be the two-sided ideal generated by  $\mathbf{a}$ .

If the elements  $a_i$  are contained in a central subring  $C \subseteq A$  such that  $A$  is flat over  $C$ , and moreover the sequence  $\mathbf{a}$  is weakly proregular in  $C$  (Definition 4.1), then we are basically back in Example 3.10, and the torsion class  $\mathsf{T}_{\mathbf{a}}$  is weakly stable.

If  $n = 1$  and  $a_1$  is a regular normalizing element, then  $\mathsf{T}_{\mathbf{a}}$  is weakly stable. The proof is very much like the commutative case, as in Section 4. See [Vs2].

A more complicated situation is when the sequence  $\mathbf{a}$  is regular normalizing in  $A$ , and  $n > 1$ . We believe that in this case too the torsion class  $\mathsf{T}_{\mathbf{a}}$  is weakly stable, but this has not been verified.

**Example 3.15.** Here is a graded variant, due to Van den Bergh in his seminal paper [VdB]. Let  $\mathbb{K}$  be a field, and let  $A = \bigoplus_{i \geq 0} A_i$  be a noetherian connected graded  $\mathbb{K}$ -ring. Recall that “connected” means that  $A_0 = \mathbb{K}$ , and each  $A_i$  is a finitely generated  $\mathbb{K}$ -module. The augmentation ideal of  $A$  is  $\mathfrak{m} := \bigoplus_{i > 0} A_i$ . We view  $\mathbb{K}$  as an  $A$ -bimodule using the isomorphism  $A/\mathfrak{m} \cong \mathbb{K}$ .

In this example we use the notation  $\mathsf{M}(A) := \mathsf{GrMod} A$ , the abelian category of *graded* left  $A$ -modules (with degree preserving homomorphisms). Inside  $\mathsf{M}(A)$  there is the torsion class  $\mathsf{T}_{\mathfrak{m}}$  of  $\mathfrak{m}$ -torsion modules; cf. Definition 3.3. Let  $A^* := \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$ , the graded dual of  $A$ . It is both  $\mathfrak{m}$ -torsion and injective in the category  $\mathsf{M}(A)$ . Any  $\mathfrak{m}$ -torsion module  $M$  can be embedded in a sufficiently large direct sum  $I = \bigoplus_{x \in X} A^*$ . Because  $A$  is noetherian, this module  $I$  is injective in the category  $\mathsf{M}(A)$ . This implies that  $\mathsf{T}_{\mathfrak{m}}$  is a stable torsion class. The noetherian property also guarantees that  $\mathsf{T}_{\mathfrak{m}}$  is quasi-compact.

Now let us consider the connected graded ring  $A^{\mathrm{en}} := A \otimes_{\mathbb{K}} A^{\mathrm{op}}$ . This is often not a noetherian ring. However, it was proved in [VdB] that  $A^{\mathrm{en}}$  is an *Ext-finite* ring, namely for any  $q$  the  $\mathbb{K}$ -module  $\mathrm{Ext}_{A^{\mathrm{en}}}^q(\mathbb{K}, \mathbb{K})$  is finitely generated.

Inside the abelian category  $\mathsf{M}(A^{\mathrm{en}})$  we have the torsion class  $\mathsf{T}_{\mathfrak{m}}$ , consisting of the bimodules  $M$  that are  $\mathfrak{m}$ -torsion as left modules. There is a mirror-image torsion class  $\mathsf{T}_{\mathfrak{m}^{\mathrm{op}}} \subseteq \mathsf{M}(A^{\mathrm{en}})$ , consisting of the bimodules  $M$  that are  $\mathfrak{m}^{\mathrm{op}}$ -torsion, or in other words, that are  $\mathfrak{m}$ -torsion as right modules. Van den Bergh proved (not in this terminology of course) that the torsion classes  $\mathsf{T}_{\mathfrak{m}}, \mathsf{T}_{\mathfrak{m}^{\mathrm{op}}} \subseteq \mathsf{M}(A^{\mathrm{en}})$  are weakly stable and quasi-compact. Furthermore, he proved that if  $\mathsf{T}_{\mathfrak{m}} \subseteq \mathsf{M}(A)$  is finite dimensional, then so is  $\mathsf{T}_{\mathfrak{m}} \subseteq \mathsf{M}(A^{\mathrm{en}})$ . The same is true for the torsion class  $\mathsf{T}_{\mathfrak{m}^{\mathrm{op}}}$ .

One of the main technical result in [VdB] is this (in our terminology): assume that  $\mathsf{T}_{\mathfrak{m}}$  and  $\mathsf{T}_{\mathfrak{m}^{\mathrm{op}}}$  are both finite dimensional torsion classes. If  $M \in \mathsf{D}(A^{\mathrm{en}})$  satisfies  $\mathrm{R}\Gamma_{\mathsf{T}_{\mathfrak{m}}}(M) \in \mathsf{T}_{\mathfrak{m}^{\mathrm{op}}}$  and  $\mathrm{R}\Gamma_{\mathsf{T}_{\mathfrak{m}^{\mathrm{op}}}}(M) \in \mathsf{T}_{\mathfrak{m}}$ , then

$$\mathrm{R}\Gamma_{\mathsf{T}_{\mathfrak{m}^{\mathrm{op}}}}(M) \cong \mathrm{R}\Gamma_{\mathsf{T}_{\mathfrak{m}}}(M)$$

in  $\mathsf{D}(A^{\mathrm{en}})$ .

In Theorem 0.7 we give an analogous result for rings  $A$  and  $B$  that are flat over a central base ring  $\mathbb{K}$ , for torsion classes  $\mathsf{T} \subseteq \mathsf{M}(A)$  and  $\mathsf{S}^{\mathrm{op}} \subseteq \mathsf{M}(B^{\mathrm{op}})$ , and for a complex  $M \in \mathsf{D}(A \otimes_{\mathbb{K}} B^{\mathrm{op}})$ .

#### 4. WEAKLY PROREGULAR IDEALS IN COMMUTATIVE RINGS

In this section we compare the noncommutative torsion picture from Section 3 to the commutative picture that was studied in [PSY1]. This comparison will provide motivation for the subsequent sections of our paper. Throughout this section  $A$  is a nonzero commutative ring.

We shall start by recalling the relevant definitions and facts from [PSY1]. Let  $\mathbf{a} = (a_1, \dots, a_n)$  be a sequence of elements of  $A$ . To this sequence we associate the *Koszul complex*  $K(A; \mathbf{a})$ , which is a bounded complex of free  $A$ -modules, concentrated in degrees  $-n, \dots, 0$ . The Koszul complex of a single element  $a \in A$  is

$$K(A; a) := (\cdots \rightarrow 0 \rightarrow A \xrightarrow{a} A \rightarrow 0 \rightarrow \cdots),$$

concentrated in degrees  $-1, 0$ . The Koszul complex of the sequence  $\mathbf{a}$  is

$$K(A; \mathbf{a}) := K(A; a_1) \otimes_A \cdots \otimes_A K(A; a_n).$$

Thus  $K(A; \mathbf{a})^0 = A$ , and  $K(A; \mathbf{a})^{-1}$  is a free  $A$ -module of rank  $n$ .

Note that  $K(A; \mathbf{a})$  is actually a commutative DG (differential graded) ring, and there is a canonical ring isomorphism

$$H^0(K(A; \mathbf{a})) \cong A/\mathfrak{a},$$

where  $\mathfrak{a} \subseteq A$  is the ideal generated by the sequence  $\mathbf{a}$ .

For any  $i \geq 1$  let  $\mathbf{a}^i$  be the sequence  $(a_1^i, \dots, a_n^i)$ . There is a corresponding Koszul complex  $K(A; \mathbf{a}^i)$ . For  $j \geq i$  there is a DG ring homomorphism

$$K(A; \mathbf{a}^j) \rightarrow K(A; \mathbf{a}^i),$$

that in degree 0 is the identity of  $A$ , and in degree  $-1$  is multiplication by the sequence  $\mathbf{a}^{j-i}$ . In this way the collection of complexes of  $A$ -modules (or of DG rings)  $\{K(A; \mathbf{a}^i)\}_{i \geq 1}$  becomes an inverse system.

**Definition 4.1.** A finite sequence  $\mathbf{a} = (a_1, \dots, a_n)$  in  $A$  is called *weakly proregular* if for every  $p < 0$ , the inverse system of  $A$ -modules

$$\{H^p(K(A; \mathbf{a}^i))\}_{i \geq 1}$$

satisfies the trivial Mittag-Leffler condition; namely for every  $i$  there is some  $j \geq i$  such that the homomorphism

$$H^p(K(A; \mathbf{a}^j)) \rightarrow H^p(K(A; \mathbf{a}^i))$$

is zero.

**Definition 4.2.** An ideal  $\mathfrak{a} \subseteq A$  is called *weakly proregular* if it is generated by some weakly proregular sequence  $\mathbf{a}$ .

Definition 4.1 was first considered in [LC]. The name “weakly proregular” was given in [AJL, Correction].

Here are some facts about weak proregularity, quoted from [PSY1]. The *dual Koszul complex* of  $\mathbf{a}$  is the complex

$$K^\vee(A; \mathbf{a}) := \text{Hom}_A(K(A; \mathbf{a}), A).$$

The collection of complexes  $\{K^\vee(A; \mathbf{a}^i)\}_{i \geq 1}$  is a direct system, and in the limit we obtain the *infinite dual Koszul complex*

$$K_\infty^\vee(A; \mathbf{a}) := \lim_{i \rightarrow} K^\vee(A; \mathbf{a}^i).$$

It is a complex of flat  $A$ -modules, concentrated in degrees  $0, \dots, n$ . The infinite dual Koszul complex of a single element  $a \in A$  looks like this:

$$(4.3) \quad K_\infty^\vee(A; a) \cong (\cdots \rightarrow 0 \rightarrow A \rightarrow A[a^{-1}] \rightarrow 0 \rightarrow \cdots),$$

where  $A$  sits in degree 0, and the differential  $A \rightarrow A[a^{-1}]$  is the ring homomorphism. For the sequence  $\mathbf{a}$  we have an isomorphism of complexes

$$(4.4) \quad K_\infty^\vee(A; \mathbf{a}) \cong K_\infty^\vee(A; a_1) \otimes_A \cdots \otimes_A K_\infty^\vee(A; a_n).$$

Let  $\mathfrak{a}$  be the ideal generated by the sequence  $\mathbf{a}$ . From formulas (4.3) and (4.4) it is clear that for any  $A$ -module  $M$  there is a canonical isomorphism

$$(4.5) \quad H^0(K_\infty^\vee(A; \mathbf{a}) \otimes_A M) \cong \Gamma_{\mathfrak{a}}(M).$$

**Theorem 4.6** ([PSY1, Theorem 3.24]). *Let  $\mathbf{a}$  be a finite sequence in  $A$ . The following two conditions are equivalent:*

- (i) *The sequence  $\mathbf{a}$  is weakly proregular.*
- (ii) *For any injective  $A$ -module  $I$  and any positive integer  $p$ , the module*

$$H^p(K_\infty^\vee(A; \mathbf{a}) \otimes_A I)$$

*is zero.*

**Theorem 4.7** ([PSY1, Theorem 3.34]). *If  $A$  is noetherian, then any finite sequence  $\mathbf{a}$  in  $A$  is weakly proregular.*

Theorems 4.6 and 4.7 already appeared in [LC].

**Theorem 4.8** ([PSY1, Corollary 5.4]). *Let  $\mathbf{a}$  and  $\mathbf{b}$  be finite sequences in  $A$ , and let  $\mathfrak{a}$  and  $\mathfrak{b}$  be the respective ideals that they generate. If the sequence  $\mathbf{a}$  is weakly proregular, and if there is equality  $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$ , then the sequence  $\mathbf{b}$  is weakly proregular.*

Suppose  $\mathfrak{a}$  is a finitely generated ideal in  $A$ . The  $\mathfrak{a}$ -torsion functor  $\Gamma_{\mathfrak{a}}$  was discussed in Section 3. It is part of an idempotent copointed additive functor  $(\Gamma_{\mathfrak{a}}, \sigma)$  on  $\mathbf{M}(A)$ . As explained in Section 2, there is a corresponding copointed triangulated functor  $(R\Gamma_{\mathfrak{a}}, \sigma^R)$  on  $\mathbf{D}(A)$ . A complex  $M \in \mathbf{D}(A)$  is called *cohomologically  $\mathfrak{a}$ -torsion* if  $\sigma_M^R : R\Gamma_{\mathfrak{a}}(M) \rightarrow M$  is an isomorphism. The full subcategory of  $\mathbf{D}(A)$  on the cohomologically  $\mathfrak{a}$ -torsion complexes is denoted by  $\mathbf{D}(A)_{\mathfrak{a}\text{-tor}}$ . It is a triangulated category.

The  $\mathfrak{a}$ -adic completion is the additive functor  $\Lambda_{\mathfrak{a}} : \mathbf{M}(A) \rightarrow \mathbf{M}(A)$  defined by

$$\Lambda_{\mathfrak{a}}(M) := \varprojlim_i (M/\mathfrak{a}^i \cdot M).$$

There is a morphism of functors  $\tau : \text{Id} \rightarrow \Lambda_{\mathfrak{a}}$ , and the pair  $(\Lambda_{\mathfrak{a}}, \tau)$  is an idempotent pointed additive functor on  $\mathbf{M}(A)$ . There is a corresponding pointed triangulated functor  $(L\Lambda_{\mathfrak{a}}, \tau^L)$  on  $\mathbf{D}(A)$ . A complex  $M \in \mathbf{D}(A)$  is called *cohomologically  $\mathfrak{a}$ -adically complete* if  $\tau_M^L : M \rightarrow L\Lambda_{\mathfrak{a}}(M)$  is an isomorphism. The full subcategory of  $\mathbf{D}(A)$  on the cohomologically  $\mathfrak{a}$ -adically complete complexes is denoted by  $\mathbf{D}(A)_{\mathfrak{a}\text{-com}}$ . It is a triangulated category.

It turns out that if  $\mathfrak{a}$  is weakly proregular, then the triangulated (co)pointed functors  $(R\Gamma_{\mathfrak{a}}, \sigma^R)$  and  $(L\Lambda_{\mathfrak{a}}, \tau^L)$  are idempotent. This is partly proved in [PSY1, Corollary 4.30 and Proposition 7.10] – these results only state that  $\sigma_{R\Gamma_{\mathfrak{a}}(M)}^R$  and  $\tau_{L\Lambda_{\mathfrak{a}}(M)}^L$  are isomorphisms. But the full idempotence is an immediate consequence of [PSY1, Lemma 7.9]. Moreover, the MGM Equivalence (Theorem 0.1 in the Introduction) is proved in [PSY1].

If  $\mathbf{a}$  is a finite sequence of elements that generates  $\mathfrak{a}$  (so it is a weakly proregular sequence by Theorem 4.8), then there is a better representative for  $R\Gamma_{\mathfrak{a}}(A)$  in  $\mathbf{D}(A)$  than the infinite Koszul complex  $K_\infty^\vee(A; \mathbf{a})$  – it is the *telescope complex*  $\text{Tel}(A; \mathbf{a})$ .

This is a bounded complex of countable rank free  $A$  modules, so in particular it is  $K$ -projective. The derived functors take these nice forms:

$$R\Gamma_{\mathfrak{a}}(M) \cong \mathrm{Tel}(A; \mathfrak{a}) \otimes_A M$$

and

$$L\Lambda_{\mathfrak{a}}(M) \cong \mathrm{Hom}_A(\mathrm{Tel}(A; \mathfrak{a}), M).$$

In the terminology of the Section 6,  $\mathrm{Tel}(A; \mathfrak{a})$  is an idempotent copointed object in  $D(A)$ . (In fact it is an idempotent copointed object in the homotopy category  $K(A)$ .)

Here are several examples of weakly proregular ideals.

**Example 4.9.** Let  $A$  be a commutative ring. If  $\mathfrak{a} = (a_1, \dots, a_n)$  is a regular sequence in  $A$ , then it is weakly proregular.

**Example 4.10.** Let  $B$  be a commutative ring that is not noetherian. Let  $A := B[t]$ , the polynomial ring in one variable, and let  $\mathfrak{a} \subseteq A$  be the ideal generated by  $t$ . Then  $A$  is not noetherian, but the ideal  $\mathfrak{a}$  is weakly proregular (by Example 4.9).

**Example 4.11.** Let  $\mathbb{K}$  be a field of characteristic 0, and let  $\mathbb{K}[[t_1]]$  and  $\mathbb{K}[[t_2]]$  be the power series rings. Let  $A$  be the ring

$$A := \mathbb{K}[[t_1]] \otimes_{\mathbb{K}} \mathbb{K}[[t_2]],$$

and let  $\mathfrak{a} \subseteq A$  be the ideal generated by  $t_1$  and  $t_2$ . As shown in [Ye5, Theorem 0.9] the ring  $A$  is not noetherian. By Example 4.9 the ideal  $\mathfrak{a}$  is weakly proregular. What is remarkable in this example is that the  $\mathfrak{a}$ -adic completion  $\hat{A}$  is noetherian, and it is not flat over  $A$ .

Now for the main result in this section. Recall that a finitely generated ideal  $\mathfrak{a} \subseteq A$  gives rise to a torsion class  $\mathsf{T}_{\mathfrak{a}} \subseteq \mathsf{M}(A)$ ; see Definition 3.3. If  $\mathfrak{b} \subseteq A$  is another finitely generated ideal, and  $\mathsf{T}_{\mathfrak{a}} = \mathsf{T}_{\mathfrak{b}}$ , then  $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$ . Theorem 4.8 tells us that weak proregularity is a property of the torsion class  $\mathsf{T}_{\mathfrak{a}}$ . The next theorem goes one step more: it lets us characterize weak proregularity in terms of a “noncommutative” property of  $\mathsf{T}_{\mathfrak{a}}$ .

**Theorem 4.12.** *Let  $A$  be a commutative ring, let  $\mathfrak{a}$  be a finite sequence of elements of  $A$ , and let  $\mathfrak{a}$  be the ideal generated by  $\mathfrak{a}$ . The following two conditions are equivalent:*

- (i) *The sequence  $\mathfrak{a}$  is weakly proregular.*
- (ii) *The torsion class  $\mathsf{T}_{\mathfrak{a}}$  is weakly stable.*

We need a lemma first.

**Lemma 4.13.** *Let  $M$  be any  $A$ -module, and consider the complex*

$$K := K_{\infty}^{\vee}(A; \mathfrak{a}) \otimes_A M.$$

- (1) *For any  $p \geq 1$  we have  $\Gamma_{\mathfrak{a}}(K^p) = 0$ , and the module  $K^p$  is right  $\Gamma_{\mathfrak{a}}$ -acyclic.*
- (2) *For any  $p$  the module  $H^p(K)$  is  $\mathfrak{a}$ -torsion.*

*Proof.* (1) Let us write the sequence in full:  $\mathfrak{a} = (a_1, \dots, a_n)$ . Fix  $p \geq 1$ . By formulas (4.3) and (4.4) we know that there is an isomorphism  $K^p \cong \bigoplus_{j=1}^n K_j^p$ , where  $K_j^p = A[a_j^{-1}] \otimes_A L_j$  for certain  $A$ -modules  $L_j$ . For any  $q$  the action of the element  $a_j$  on the module  $R^q\Gamma_{\mathfrak{a}}(K_j^p)$  is bijective; yet the action of  $a_j$  on any finitely generated submodule of  $R^q\Gamma_{\mathfrak{a}}(K_j^p)$  is nilpotent. This implies that  $R^q\Gamma_{\mathfrak{a}}(K_j^p) = 0$

for all  $q$ . We see that  $\Gamma_{\mathfrak{a}}(K_j^p) = 0$ , and that  $K_j^p$  is right  $\Gamma_{\mathfrak{a}}$ -acyclic. Hence the same is true for  $K^p$ .

(2) By formulas (4.3) and (4.4), for any  $j$  the complex

$$A[a_j^{-1}] \otimes_A K_{\infty}^{\vee}(A; \mathfrak{a})$$

is contractible. Therefore for every  $p$

$$A[a_j^{-1}] \otimes_A H^p(K) \cong H^p(A[a_j^{-1}] \otimes_A K) = 0.$$

In view of equation (4.5), this shows that the module  $H^p(K)$  is  $a_j$ -torsion for all  $j$ . Hence it is  $\mathfrak{a}$ -torsion.  $\square$

*Proof of Theorem 4.12.* (i)  $\Rightarrow$  (ii): By [PSY1, Corollary 4.30], for any  $M \in \mathcal{D}(A)$  the morphism

$$(4.14) \quad \sigma_{R\Gamma_{\mathfrak{a}}(M)}^R : R\Gamma_{\mathfrak{a}}(R\Gamma_{\mathfrak{a}}(M)) \rightarrow R\Gamma_{\mathfrak{a}}(M)$$

in  $\mathcal{D}(A)$  is an isomorphism. Take any injective  $A$ -module  $I$ . Then  $R\Gamma_{\mathfrak{a}}(I) \cong \Gamma_{\mathfrak{a}}(I)$ , and formula (4.14) says that  $R\Gamma_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(I)) \cong \Gamma_{\mathfrak{a}}(I)$  in  $\mathcal{D}(A)$ . Therefore

$$R^q\Gamma_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(I)) \cong H^q(R\Gamma_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(I))) \cong H^q(\Gamma_{\mathfrak{a}}(I)),$$

which is zero for all  $q > 0$ . We see that the module  $\Gamma_{\mathfrak{a}}(I)$  is right  $\Gamma_{\mathfrak{a}}$ -acyclic (Definition 1.1). We conclude that the functor  $\Gamma_{\mathfrak{a}}$  is weakly stable (Definition 2.1), and that the torsion class  $\mathcal{T}_{\mathfrak{a}}$  is weakly stable (Definition 3.4).

(ii)  $\Rightarrow$  (i): According to Theorem 4.6, it suffices to show that for any injective  $A$ -module  $I$  and any positive integer  $p$ , the module  $H^p(K_{\infty}^{\vee}(A; \mathfrak{a}) \otimes_A I)$  is zero.

The proof is by contradiction. Assume, for the sake of contradiction, that for some injective  $A$ -module  $I$ , and for some  $p \geq 1$ , we have

$$H^p(K_{\infty}^{\vee}(A; \mathfrak{a}) \otimes_A I) \neq 0.$$

Let us write

$$K := K_{\infty}^{\vee}(A; \mathfrak{a}) \otimes_A I.$$

We may assume that  $p$  is minimal:  $H^p(K) \neq 0$ , but  $H^q(K) = 0$  for all  $1 \leq q < p$ . Thus we have an exact sequence

$$(4.15) \quad 0 \rightarrow \Gamma_{\mathfrak{a}}(I) \rightarrow K^0 \rightarrow \cdots \rightarrow K^{p-1} \rightarrow K^p.$$

Since the module  $K^0 \cong I$  is injective, it is right  $\Gamma_{\mathfrak{a}}$ -acyclic. For  $1 \leq q \leq p$  the module  $K^q$  is right  $\Gamma_{\mathfrak{a}}$ -acyclic by Lemma 4.13(1).

Let us now write  $J^q := K^q$  for  $q \leq p$ , and continue the exact sequence (4.15), with its new notation, to an exact sequence

$$(4.16) \quad 0 \rightarrow \Gamma_{\mathfrak{a}}(I) \rightarrow J^0 \rightarrow \cdots \rightarrow J^{p-1} \rightarrow J^p \rightarrow J^{p+1} \rightarrow \cdots$$

in which the modules  $J^q$ , for  $q > p$ , are also right  $\Gamma_{\mathfrak{a}}$ -acyclic (e.g. we can take injective  $A$ -modules). So we get a complex  $J$  of right  $\Gamma_{\mathfrak{a}}$ -acyclic modules that is concentrated in nonnegative degrees, and a quasi-isomorphism  $\Gamma_{\mathfrak{a}}(I) \rightarrow J$ . By Lemma 1.5 we know that  $J$  is a right  $\Gamma_{\mathfrak{a}}$ -acyclic complex, and thus

$$(4.17) \quad R^q\Gamma_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(I)) \cong H^q(\Gamma_{\mathfrak{a}}(J))$$

for all  $q$ .

There is an embedding

$$H^p(K) \subseteq \text{Coker}(K^{p-1} \rightarrow K^p) = \text{Coker}(J^{p-1} \rightarrow J^p).$$

On the other hand, due to the exactness of (4.16), we have

$$\text{Coker}(J^{p-1} \rightarrow J^p) \cong \text{Ker}(J^{p+1} \rightarrow J^{p+2}).$$

By Lemma 4.13(2) the module  $H^p(K)$  is  $\mathfrak{a}$ -torsion, and therefore we get an embedding

$$H^p(K) \subseteq \Gamma_{\mathfrak{a}}(\text{Ker}(J^{p+1} \rightarrow J^{p+2})).$$

But

$$\Gamma_{\mathfrak{a}}(\text{Ker}(J^{p+1} \rightarrow J^{p+2})) \cong \text{Ker}(\Gamma_{\mathfrak{a}}(J^{p+1}) \rightarrow \Gamma_{\mathfrak{a}}(J^{p+2})).$$

Since  $H^p(K) \neq 0$ , we conclude that

$$(4.18) \quad \text{Ker}(\Gamma_{\mathfrak{a}}(J^{p+1}) \rightarrow \Gamma_{\mathfrak{a}}(J^{p+2})) \neq 0.$$

The module  $\Gamma_{\mathfrak{a}}(I)$  is right  $\Gamma_{\mathfrak{a}}$ -acyclic; this is because  $I$  is injective and  $\mathsf{T}_{\mathfrak{a}}$  is weakly stable. From equation (4.17) we conclude that

$$(4.19) \quad H^q(\Gamma_{\mathfrak{a}}(J)) = 0 \quad \text{for all } q > 0.$$

Also  $\Gamma_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(I)) = \Gamma_{\mathfrak{a}}(I)$ . Therefore the sequence

$$(4.20) \quad 0 \rightarrow \Gamma_{\mathfrak{a}}(I) \rightarrow \Gamma_{\mathfrak{a}}(J^0) \rightarrow \Gamma_{\mathfrak{a}}(J^1) \rightarrow \cdots \rightarrow \Gamma_{\mathfrak{a}}(J^p) \rightarrow \Gamma_{\mathfrak{a}}(J^{p+1}) \rightarrow \cdots,$$

gotten from (4.16) by applying the functor  $\Gamma_{\mathfrak{a}}$ , is exact.

Finally, by Lemma 4.13(1) we know that  $\Gamma_{\mathfrak{a}}(J^q) = 0$  for all  $1 \leq q \leq p$ . So the sequence (4.20) looks like this:

$$0 \rightarrow \Gamma_{\mathfrak{a}}(I) \rightarrow \Gamma_{\mathfrak{a}}(J^0) \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow \Gamma_{\mathfrak{a}}(J^{p+1}) \rightarrow \Gamma_{\mathfrak{a}}(J^{p+2}) \rightarrow \cdots,$$

with at least one 0 occurring between  $\Gamma_{\mathfrak{a}}(J^0)$  and  $\Gamma_{\mathfrak{a}}(J^{p+1})$ . Combining this with equation (4.18) we conclude that  $H^{p+1}(\Gamma_{\mathfrak{a}}(J)) \neq 0$ . This contradicts (4.19).  $\square$

**Corollary 4.21.** *Let  $A$  be a commutative ring, and let  $\mathfrak{a}$  be a finitely generated ideal of  $A$ . If the torsion class  $\mathsf{T}_{\mathfrak{a}}$  is weakly stable, then it is also quasi-compact and finite dimensional.*

*Proof.* Let  $\mathbf{a}$  be a finite sequence, say of length  $n$ , that generates the ideal  $\mathfrak{a}$ . By the theorem we know that the sequence  $\mathbf{a}$  is weakly proregular. According to [PSY1, Corollary 3.26] there is an isomorphism of triangulated functors

$$\mathbf{R}\Gamma_{\mathfrak{a}} \cong \mathbf{K}_{\infty}^{\vee}(A; \mathbf{a}) \otimes_A -.$$

This shows that the functor  $\mathbf{R}\Gamma_{\mathfrak{a}}$  is quasi-compact, and that its cohomological dimension is at most  $n$ . Since the triangulated functor  $\mathbf{R}\Gamma_{\mathfrak{a}}$  is quasi-compact, then so are the functors  $\mathbf{R}^q\Gamma_{\mathfrak{a}}$ . And  $\mathbf{R}^q\Gamma_{\mathfrak{a}} = 0$  for  $q > n$ .  $\square$

**Remark 4.22.** Corollary 4.21 is false for general torsion classes. Indeed, there is an example of a noncommutative ring  $A$ , and a weakly stable torsion class  $\mathsf{T} \subseteq \mathsf{M}(A)$ , that is not finite dimensional nor quasi-compact. This example will be discussed in the upcoming paper [Vs2].

## 5. DERIVED CATEGORIES OF BIMODULES

In this section we explain how to form derived tensor and Hom functors between derived categories of bimodules over noncommutative rings, and study some of their properties. We do this in a somewhat narrow setting: we work over a commutative base ring  $\mathbb{K}$ , and some of the rings are assumed to be flat over it. See Remark 5.22 regarding the use of DG rings to remove the flatness assumption.

From now on in the paper we adopt the following convention.

**Convention 5.1.** There is a fixed nonzero commutative base ring  $\mathbb{K}$ . All additive operations and categories are by default  $\mathbb{K}$ -linear. All rings and bimodules are central over  $\mathbb{K}$ . We use the abbreviation  $\otimes$  for  $\otimes_{\mathbb{K}}$ . By default all modules are left modules. We say that  $A$  is a *flat ring* if  $A$  is flat as a  $\mathbb{K}$ -module.

**Example 5.2.** If  $\mathbb{K}$  is a field, then any central  $\mathbb{K}$ -ring  $A$  is flat. If  $\mathbb{K} = \mathbb{Z}$ , then any ring  $A$  is  $\mathbb{K}$ -central, and  $A$  is flat if and only if it is torsion free (as an abelian group).

Let  $A$  and  $B$  be rings. Recall that the *opposite* ring of  $A$  is the ring  $A^{\text{op}}$ , that has the same underlying  $\mathbb{K}$ -module structure as  $A$ , but the multiplication is

$$a_1 \cdot^{\text{op}} a_2 := a_2 \cdot a_1$$

for  $a_1, a_2 \in A$ . The tensor product  $A \otimes B$  is a ring with the usual multiplication

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) := (a_1 \cdot a_2) \otimes (b_1 \cdot b_2)$$

for  $a_k \in A$  and  $b_k \in B$ . The *enveloping* ring of  $A$  is the ring  $A^{\text{en}} := A \otimes A^{\text{op}}$ . We identify right  $B$ -modules with left  $B^{\text{op}}$ -modules, and  $A$ - $B$ -bimodules with left modules over  $A \otimes B^{\text{op}}$ .

Given  $M \in \mathbf{M}(A^{\text{op}})$  and  $N \in \mathbf{M}(A)$ , the tensor product  $M \otimes_A N$  is defined; and it is a  $\mathbb{K}$ -module. Note that  $(A^{\text{op}})^{\text{op}} = A$ . It is a bit confusing, but actually easy to check, that there is a canonical isomorphism

$$(5.3) \quad N \otimes_{A^{\text{op}}} M \cong M \otimes_A N$$

in  $\mathbf{M}(\mathbb{K})$ , as quotients of  $N \otimes M \cong M \otimes N$ .

A homomorphism of rings  $f : A \rightarrow B$  induces a forgetful functor

$$\text{Rest}_f : \mathbf{M}(B) \rightarrow \mathbf{M}(A)$$

that we call *restriction*, which is exact. The restriction functor extends to complexes, and to a triangulated functor

$$(5.4) \quad \text{Rest}_f : \mathbf{D}(B) \rightarrow \mathbf{D}(A).$$

**Proposition 5.5.** *Let  $f : A \rightarrow B$  be a ring homomorphism. The functor  $\text{Rest}_f$  is conservative. Namely a morphism  $\phi : M \rightarrow N$  in  $\mathbf{D}(B)$  is an isomorphism if and only if the morphism*

$$\text{Rest}_f(\phi) : \text{Rest}_f(M) \rightarrow \text{Rest}_f(N)$$

*in  $\mathbf{D}(A)$  is an isomorphism.*

*Proof.* The morphism  $\phi$  is an isomorphism in  $\mathbf{D}(B)$  if and only if the  $\mathbb{K}$ -module homomorphisms  $H^p(\phi) : H^p(M) \rightarrow H^p(N)$  are isomorphisms for all  $p$ . But  $H^p(\phi) = H^p(\text{Rest}_f(\phi))$  as  $\mathbb{K}$ -module homomorphisms.  $\square$



Given rings  $A$  and  $B$ , there is a canonical ring homomorphism  $A \rightarrow A \otimes B$ , that sends  $a \mapsto a \otimes 1_B$ . The corresponding forgetful functor is denoted by

$$(5.6) \quad \text{Rest}_A : D(A \otimes B) \rightarrow D(A).$$

**Definition 5.7.** Let  $A$  and  $B$  be rings, and let  $M \in C(A \otimes B)$ . If  $\text{Rest}_A(M) \in C(A)$  is  $K$ -flat (resp.  $K$ -injective, resp.  $K$ -projective), then we say that  $M$  is  $K$ -flat (resp.  $K$ -injective, resp.  $K$ -projective) over  $A$ .

**Lemma 5.8.** Let  $A$  and  $B$  be rings, and assume  $B$  is flat.

- (1) If  $P \in C(A \otimes B)$  is  $K$ -flat, then  $P$  is  $K$ -flat over  $A$ .
- (2) If  $I \in C(A \otimes B)$  is  $K$ -injective, then  $I$  is  $K$ -injective over  $A$ .

*Proof.* These are direct consequences of the identities

$$M \otimes_A P \cong (M \otimes B) \otimes_{A \otimes B} P$$

and

$$\text{Hom}_A(N, I) \cong \text{Hom}_{A \otimes B}(N \otimes B, I)$$

for  $M \in M(A^{\text{op}})$  and  $N \in M(A)$ .  $\square$

**Remark 5.9.** If  $B$  happens to be a projective module over  $\mathbb{K}$ , then the restriction to  $A$  of a  $K$ -projective complex in  $C(A \otimes B)$  is  $K$ -projective in  $C(A)$ . The proof is similar. We will not need this fact here.

**Proposition 5.10.** Let  $A$ ,  $B$  and  $C$  be rings, and assume  $C$  is flat.

- (1) The bifunctor

$$- \otimes_B - : M(A \otimes B^{\text{op}}) \times M(B \otimes C^{\text{op}}) \rightarrow M(A \otimes C^{\text{op}})$$

has a left derived bifunctor

$$- \otimes_B^L - : D(A \otimes B^{\text{op}}) \times D(B \otimes C^{\text{op}}) \rightarrow D(A \otimes C^{\text{op}}).$$

- (2) Given  $M \in C(A \otimes B^{\text{op}})$  and  $N \in C(B \otimes C^{\text{op}})$ , such that  $M$  is  $K$ -flat over  $B^{\text{op}}$  or  $N$  is  $K$ -flat over  $B$ , the morphism

$$\xi_{M,N}^L : M \otimes_B^L N \rightarrow M \otimes_B N$$

in  $D(A \otimes C^{\text{op}})$  is an isomorphism.

- (3) Suppose we are given ring homomorphisms  $A' \rightarrow A$  and  $C' \rightarrow C$ , such that  $C'$  is flat. Then the diagram

$$\begin{array}{ccc} D(A \otimes B^{\text{op}}) \times D(B \otimes C^{\text{op}}) & \xrightarrow{- \otimes_B^L -} & D(A \otimes C^{\text{op}}) \\ \text{Rest} \times \text{Rest} \downarrow & & \downarrow \text{Rest} \\ D(A' \otimes B^{\text{op}}) \times D(B \otimes C'^{\text{op}}) & \xrightarrow{- \otimes_B^L -} & D(A' \otimes C'^{\text{op}}) \end{array}$$

is commutative up to an isomorphism of triangulated bifunctors.

- (4) Suppose  $D$  is another flat ring. Then there is an isomorphism

$$(- \otimes_B^L -) \otimes_C^L - \cong - \otimes_B^L (- \otimes_C^L -)$$

of triangulated trifunctors

$$D(A \otimes B^{\text{op}}) \times D(B \otimes C^{\text{op}}) \times D(C \otimes D^{\text{op}}) \rightarrow D(A \otimes D^{\text{op}}).$$

Note that item (3) includes the cases  $A' = \mathbb{K}$  and  $C' = \mathbb{K}$ .

*Proof.* (1) The bifunctor  $-\otimes_B-$  induces a triangulated bifunctor

$$-\otimes_B- : \mathcal{K}(A \otimes B^{\text{op}}) \times \mathcal{K}(B \otimes C^{\text{op}}) \rightarrow \mathcal{K}(A \otimes C^{\text{op}})$$

on the homotopy categories, in the obvious way. By Lemma 5.8(1), any DG bimodule  $N \in \mathcal{C}(B \otimes C^{\text{op}})$  admits a quasi-isomorphism  $\zeta_N : \tilde{N} \rightarrow N$ , where  $\tilde{N} \in \mathcal{C}(B \otimes C^{\text{op}})$  is K-flat over  $B$  (e.g. we can take  $\tilde{N}$  to be K-projective in  $\mathcal{C}(B \otimes C^{\text{op}})$ ). Let us define the object

$$M \otimes_B^{\mathbf{L}} N := M \otimes_B \tilde{N} \in \mathcal{D}(A \otimes C^{\text{op}}),$$

with the morphism

$$\xi_{M,N}^{\mathbf{L}} := \text{id}_M \otimes_B \zeta_N : M \otimes_B^{\mathbf{L}} N \rightarrow M \otimes_B N.$$

The pair  $(-\otimes_B^{\mathbf{L}}-, \xi^{\mathbf{L}})$  is a left derived bifunctor of  $-\otimes_B-$ .

(2) Under either assumption the homomorphism  $\text{id}_M \otimes_B \zeta_N$  is a quasi-isomorphism.

(3) The resolutions  $\zeta_N : \tilde{N} \rightarrow N$  from item (1) become resolutions

$$\text{Rest}(\zeta_N) : \text{Rest}(\tilde{N}) \rightarrow \text{Rest}(N)$$

in  $\mathcal{C}(B \otimes C'^{\text{op}})$  that are K-flat over  $B$ . And for  $M \in \mathcal{C}(A \otimes B^{\text{op}})$  there is an obvious isomorphism

$$\text{Rest}(M) \otimes_B \text{Rest}(\tilde{N}) \cong \text{Rest}(M \otimes_B \tilde{N})$$

in  $\mathcal{C}(A' \otimes C'^{\text{op}})$ .

(4) Given  $M, N, \tilde{N}$  as above and  $P \in \mathcal{C}(C \otimes D^{\text{op}})$ , we choose a quasi-isomorphism  $\zeta_P : \tilde{P} \rightarrow P$ , where  $\tilde{P} \in \mathcal{C}(C \otimes D^{\text{op}})$  is K-flat over  $C$ . A small calculation shows that  $\tilde{N} \otimes_C \tilde{P}$  is K-flat over  $B$ . The desired isomorphism

$$(M \otimes_B^{\mathbf{L}} N) \otimes_C^{\mathbf{L}} P \cong M \otimes_B^{\mathbf{L}} (N \otimes_C^{\mathbf{L}} P)$$

in  $\mathcal{D}(A \otimes D^{\text{op}})$  comes from the obvious isomorphism

$$(M \otimes_B \tilde{N}) \otimes_C \tilde{P} \cong M \otimes_B (\tilde{N} \otimes_C \tilde{P})$$

in  $\mathcal{C}(A \otimes D^{\text{op}})$ . □

A matter of notation: suppose we are given morphisms  $\phi : M' \rightarrow M$  in  $\mathcal{D}(A \otimes B^{\text{op}})$  and  $\psi : N' \rightarrow N$  in  $\mathcal{D}(B \otimes C^{\text{op}})$ . The result of applying the bifunctor  $-\otimes_B^{\mathbf{L}}-$  is the morphism

$$(5.11) \quad \phi \otimes_B^{\mathbf{L}} \psi : M' \otimes_B^{\mathbf{L}} N' \rightarrow M \otimes_B^{\mathbf{L}} N$$

in  $\mathcal{D}(A \otimes C^{\text{op}})$ .

**Proposition 5.12.** *Let  $A, B$  and  $C$  be rings, and assume  $C$  is flat.*

(1) *The bifunctor*

$$\text{Hom}_B(-, -) : \mathcal{M}(B \otimes A^{\text{op}})^{\text{op}} \times \mathcal{M}(B \otimes C^{\text{op}}) \rightarrow \mathcal{M}(A \otimes C^{\text{op}})$$

*has a right derived bifunctor*

$$\text{RHom}_B(-, -) : \mathcal{D}(B \otimes A^{\text{op}})^{\text{op}} \times \mathcal{D}(B \otimes C^{\text{op}}) \rightarrow \mathcal{D}(A \otimes C^{\text{op}}).$$

(2) *Given  $M \in \mathcal{C}(B \otimes A^{\text{op}})$  and  $N \in \mathcal{C}(B \otimes C^{\text{op}})$ , such that either  $M$  is K-projective over  $B$  or  $N$  is K-injective over  $B$ , the morphism*

$$\xi_{M,N}^{\mathbf{R}} : \text{Hom}_B(M, N) \rightarrow \text{RHom}_B(M, N)$$

*in  $\mathcal{D}(A \otimes C^{\text{op}})$  is an isomorphism.*

- (3) Suppose we are given ring homomorphisms  $A' \rightarrow A$  and  $C' \rightarrow C$ , such that  $C'$  is  $K$ -flat. Then the diagram

$$\begin{array}{ccc} \mathrm{D}(B \otimes A^{\mathrm{op}})^{\mathrm{op}} \times \mathrm{D}(B \otimes C^{\mathrm{op}}) & \xrightarrow{\mathrm{RHom}_B(-, -)} & \mathrm{D}(A \otimes C^{\mathrm{op}}) \\ \mathrm{Rest} \times \mathrm{Rest} \downarrow & & \downarrow \mathrm{Rest} \\ \mathrm{D}(B \otimes A'^{\mathrm{op}})^{\mathrm{op}} \times \mathrm{D}(B \otimes C'^{\mathrm{op}}) & \xrightarrow{\mathrm{RHom}_B(-, -)} & \mathrm{D}(A' \otimes C'^{\mathrm{op}}) \end{array}$$

is commutative up to an isomorphism of triangulated bifunctors.

- (4) Suppose  $D$  is another flat ring. Then for  $M \in \mathrm{D}(B \otimes A^{\mathrm{op}})$ ,  $N \in \mathrm{D}(C \otimes B^{\mathrm{op}})$  and  $L \in \mathrm{D}(C \otimes D^{\mathrm{op}})$  there is an isomorphism

$$\mathrm{RHom}_B(M, \mathrm{RHom}_C(N, L)) \cong \mathrm{RHom}_C(N \otimes_B^L M, L)$$

in  $\mathrm{D}(A \otimes D^{\mathrm{op}})$ . This isomorphism is functorial in the objects  $M, N, L$ .

*Proof.* (1-3) This is like the proof of Proposition 5.10, but now we rely on Lemma 5.8(2) and [Ye2, Proposition 2.6(2)].

(4) Here we choose a quasi-isomorphism  $L \rightarrow J$  in  $\mathcal{C}(C \otimes D^{\mathrm{op}})$  into a complex  $J$  that is  $K$ -injective over  $C$ , and a quasi-isomorphism  $\tilde{N} \rightarrow N$  in  $\mathcal{C}(C \otimes B^{\mathrm{op}})$  from a complex  $\tilde{N}$  that is  $K$ -flat over  $B^{\mathrm{op}}$ . A calculation shows that  $\mathrm{Hom}_C(\tilde{N}, J)$  is  $K$ -injective over  $B$ . The desired isomorphism comes from the obvious adjunction isomorphism

$$\mathrm{Hom}_B(M, \mathrm{Hom}_C(\tilde{N}, J)) \cong \mathrm{Hom}_C(\tilde{N} \otimes_B M, J)$$

in  $\mathcal{C}(A \otimes D^{\mathrm{op}})$ .  $\square$

Suppose we are given morphisms  $\phi : M \rightarrow M'$  in  $\mathrm{D}(B \otimes A^{\mathrm{op}})$  and  $\psi : N' \rightarrow N$  in  $\mathrm{D}(B \otimes C^{\mathrm{op}})$ . The result of applying the bifunctor  $\mathrm{RHom}_B(-, -)$  is the morphism

$$(5.13) \quad \mathrm{RHom}_B(\phi, \psi) : \mathrm{RHom}_B(M', N') \rightarrow \mathrm{RHom}_B(M, N)$$

in  $\mathrm{D}(A \otimes C^{\mathrm{op}})$ .

If  $A$  and  $B$  are flat rings, then according to Propositions 5.10 and 5.12 we obtain triangulated bifunctors

$$(5.14) \quad - \otimes_A^L - : \mathrm{D}(A^{\mathrm{en}}) \times \mathrm{D}(A^{\mathrm{en}}) \rightarrow \mathrm{D}(A^{\mathrm{en}}),$$

$$(5.15) \quad - \otimes_A^L - : \mathrm{D}(A^{\mathrm{en}}) \times \mathrm{D}(A \otimes B) \rightarrow \mathrm{D}(A \otimes B)$$

and

$$(5.16) \quad \mathrm{RHom}_A(-, -) : \mathrm{D}(A^{\mathrm{en}})^{\mathrm{op}} \times \mathrm{D}(A \otimes B) \rightarrow \mathrm{D}(A \otimes B).$$

**Proposition 5.17.** *Suppose  $A$  and  $B$  are flat rings.*

- (1) *The operation (5.14) is a monoidal structure on the category  $\mathrm{D}(A^{\mathrm{en}})$ , with unit object  $A$ .*
- (2) *The operation (5.15) is a left monoidal action of the monoidal category  $\mathrm{D}(A^{\mathrm{en}})$  on the category  $\mathrm{D}(A \otimes B)$ .*
- (3) *The operation (5.16) is a right monoidal action of the monoidal category  $\mathrm{D}(A^{\mathrm{en}})$  on the category  $\mathrm{D}(A \otimes B)$ .*

*Proof.* Since it is enough to check the coherence axioms for  $K$ -flat complexes in  $\mathrm{D}(A^{\mathrm{en}})$  and  $K$ -injective complexes in  $\mathrm{D}(A \otimes B)$ , these statements are immediate consequences of the corresponding coherence axioms for the ordinary tensor and Hom operations, and the adjunctions between them.  $\square$

Let us introduce the notation

$$(5.18) \quad \text{lu} : A \otimes_A^L M \xrightarrow{\sim} M$$

for the *left unitor* isomorphism in  $D(A^{\text{en}})$  or  $D(A \otimes B)$ ; and

$$(5.19) \quad \text{ru} : N \otimes_A^L A \xrightarrow{\sim} N$$

for the *right unitor* isomorphism in  $D(A^{\text{en}})$ . We shall also require the canonical isomorphism

$$(5.20) \quad \text{lcu} : M \xrightarrow{\sim} \text{RHom}_A(A, M)$$

coming from the action (5.16), that (for lack of pre-existing name) we call the *left co-unitor*.

**Remark 5.21.** Actually, in the setup of Proposition 5.17,  $D(A^{\text{en}})$  is a *biclosed monoidal category*. The two internal Hom operations are  $\text{RHom}_A(-, -)$  and  $\text{RHom}_{A^{\text{op}}}(-, -)$ . See [nLab] regarding these concepts.

**Remark 5.22.** Here is the way to handle derived categories of bimodules in the absence of flatness. The idea is to choose K-flat resolutions  $\tilde{A} \rightarrow A$  and  $\tilde{B} \rightarrow B$ , in the category  $\text{DGR}^{\leq 0}_{/\text{ce}} \mathbb{K}$  of nonpositive central DG  $\mathbb{K}$ -rings. See [Ye2]. Then the derived category of  $A$ - $B$ -bimodules is the triangulated category  $D(\tilde{A} \otimes \tilde{B}^{\text{op}})$ , the derived category of DG modules over the DG ring  $\tilde{A} \otimes \tilde{B}^{\text{op}}$ . Propositions 5.10, 5.12 and 5.17 have straightforward extensions to the DG case.

The fact that the category  $D(\tilde{A} \otimes \tilde{B}^{\text{op}})$  is independent of the resolutions (up to a canonical equivalence of triangulated categories) is not easy to prove. This will be done in the future paper [Ye3]; see also the lecture notes [Ye4].

## 6. IDEMPOTENT COPOINTED OBJECTS

Recall that we are working over a commutative base ring  $\mathbb{K}$ , and Convention 5.1 is in force. From this section onwards we also assume the following convention:

**Convention 6.1.** The rings  $A$  and  $B$  are flat over  $\mathbb{K}$ .

Consider the enveloping ring  $A^{\text{en}} = A \otimes A^{\text{op}}$  of  $A$ . According to Proposition 5.17 the triangulated category  $D(A^{\text{en}})$  has a monoidal structure  $- \otimes_A^L -$ , with unit object  $A$ . There are also a left monoidal action  $- \otimes_A^L -$ , and a right monoidal action  $\text{RHom}_A(-, -)$ , of  $D(A^{\text{en}})$  on  $D(A \otimes B^{\text{op}})$ . Correspondingly there are unitor isomorphisms  $\text{lu}$ ,  $\text{ru}$  and  $\text{lcu}$ , explained in formulas (5.18), (5.19) and (5.20).

**Definition 6.2.**

- (1) A *copointed object* in the monoidal category  $D(A^{\text{en}})$  is a pair  $(P, \rho)$ , consisting of complex  $P \in D(A^{\text{en}})$  and a morphism  $\rho : P \rightarrow A$  in  $D(A^{\text{en}})$ .
- (2) The copointed object  $(P, \rho)$  is called *idempotent* if the morphisms

$$\text{lu} \circ (\rho \otimes_A^L \text{id}), \text{ru} \circ (\text{id} \otimes_A^L \rho) : P \otimes_A^L P \rightarrow P$$

in  $D(A^{\text{en}})$  are both isomorphisms.

**Remark 6.3.** Here is an explanation of the name “copointed”. In a monoidal category, with unit object  $A$ , a *point* of an object  $P$  is a morphism  $A \rightarrow P$ . It thus makes sense to refer to a morphism in the dual direction, say  $\rho : P \rightarrow A$ , as a *copoint* of  $P$ .

**Definition 6.4.** Let  $(P, \rho)$  be a copointed object in  $D(A^{\text{en}})$ .

- (1) Define the triangulated functors

$$F, G : D(A \otimes B^{\text{op}}) \rightarrow D(A \otimes B^{\text{op}})$$

to be

$$F := P \otimes_A^L - \quad \text{and} \quad G := \text{RHom}_A(P, -).$$

- (2) Let

$$\sigma : F \rightarrow \text{Id}_{D(A \otimes B^{\text{op}})} \quad \text{and} \quad \tau : \text{Id}_{D(A \otimes B^{\text{op}})} \rightarrow G$$

be the morphisms of triangulated functors from  $D(A \otimes B^{\text{op}})$  to itself that are induced by the morphism  $\rho : P \rightarrow A$ . Namely

$$\sigma_M : F(M) = P \otimes_A^L M \rightarrow M$$

is

$$\sigma_M := \text{lu} \circ (\rho \otimes_A^L \text{id}_M),$$

and

$$\tau_M : M \rightarrow G(M) = \text{RHom}_A(P, M)$$

is

$$\tau := \text{RHom}_A(\rho, \text{id}_M) \circ \text{lcu}.$$

We refer to  $(F, \sigma)$  and  $(G, \tau)$  as the (co)pointed triangulated functors induced by the copointed object  $(P, \rho)$ .

See formulas 5.18 and 5.20 regarding the isomorphisms  $\text{lu}$  and  $\text{lcu}$ . Item (2) of the definition is shown in the commutative diagrams below in the category  $D(A \otimes B^{\text{op}})$ .

$$(6.5) \quad \begin{array}{ccc} P \otimes_A^L M & & M \\ \rho \otimes_A^L \text{id}_M \downarrow & \searrow \sigma_M & \downarrow \text{lcu} \\ A \otimes_A^L M & \xrightarrow{\text{lu}} & M \end{array} \quad \begin{array}{ccc} M & \xrightarrow{\text{lcu}} & \text{RHom}_A(A, M) \\ & \searrow \tau_M & \downarrow \text{RHom}_A(\rho, \text{id}_M) \\ & & \text{RHom}_A(P, M) \end{array}$$

**Definition 6.6.** Let  $(F, \sigma)$  and  $(G, \tau)$  be the copointed and pointed triangulated functors on  $D(A \otimes B^{\text{op}})$  from Definition 6.4.

- (1) We define the full triangulated subcategory  $D(A \otimes B^{\text{op}})_F$  of  $D(A \otimes B^{\text{op}})$  to be

$$D(A \otimes B^{\text{op}})_F := \{M \mid \sigma_M : F(M) \rightarrow M \text{ is an isomorphism}\}.$$

- (2) We define the full triangulated subcategory  $D(A \otimes B^{\text{op}})_G$  of  $D(A \otimes B^{\text{op}})$  to be

$$D(A \otimes B^{\text{op}})_G := \{M \mid \tau_M : M \rightarrow G(M) \text{ is an isomorphism}\}.$$

**Remark 6.7.** The notation used in the two definitions above was chosen to be consistent with that of [PSY1], where  $P$  is the *telescope complex* associated to a weakly proregular generating sequence of the ideal  $\mathfrak{a}$ . See Definition 3.8, Definition 3.11, Proposition 5.8 and Corollary 5.25 of op. cit., where the functors are  $G = L\Lambda_{\mathfrak{a}}$  and  $F = R\Gamma_{\mathfrak{a}}$ .

**Lemma 6.8.** *If the copointed object  $(P, \rho)$  is idempotent, then the copointed triangulated functor  $(F, \sigma)$  and the pointed triangulated functor  $(G, \tau)$  on  $D(A \otimes B^{\text{op}})$  are idempotent.*

*Proof.* For  $M \in \mathbf{D}(A \otimes B^{\text{op}})$  there are equalities (up to the associativity isomorphism of  $-\otimes_A^L -$ , that should be inserted in the locations marked by “ $\dagger$ ”):

$$(6.9) \quad F(\sigma_M) = \text{id}_P \otimes_A^L (\text{lu} \circ (\rho \otimes_A^L \text{id}_M)) =^\dagger (\text{ru} \circ (\text{id}_P \otimes_A^L \rho)) \otimes_A^L \text{id}_M$$

and

$$(6.10) \quad \sigma_{F(M)} = \text{lu} \circ (\rho \otimes_A^L (\text{id}_P \otimes_A^L \text{id}_M)) =^\dagger (\text{lu} \circ (\rho \otimes_A^L \text{id}_P)) \otimes_A^L \text{id}_M$$

of morphisms

$$F(F(M)) = P \otimes_A^L P \otimes_A^L M \rightarrow F(M) = P \otimes_A^L M$$

in  $\mathbf{D}(A \otimes B^{\text{op}})$ . Because both  $\rho \otimes_A^L \text{id}_P$  and  $\text{id}_P \otimes_A^L \rho$  are isomorphisms in  $\mathbf{D}(A^{\text{en}})$ , it follows that the morphisms  $F(\sigma_M)$  and  $\sigma_{F(M)}$  are isomorphisms in  $\mathbf{D}(A \otimes B^{\text{op}})$ .

There are also equalities (up to the associativity and adjunction isomorphisms of  $-\otimes_A^L -$  and  $\text{RHom}_A(-, -)$ , that should be inserted in the locations marked by “ $\ddagger$ ”):

$$(6.11) \quad \begin{aligned} G(\tau_M) &= \text{RHom}_A(\text{id}_P, \text{RHom}_A(\rho, \text{id}_M)) \circ \text{RHom}_A(\text{id}_P, \text{lcu}) \\ &=^\ddagger \text{RHom}_A(\rho \otimes_A^L \text{id}_P, \text{id}_M) \circ \text{RHom}_A(\text{lu}, \text{id}_M) \end{aligned}$$

and

$$(6.12) \quad \begin{aligned} \tau_{G(M)} &= \text{RHom}_A(\rho, \text{id}_{\text{RHom}_A(P, M)}) \circ \text{lcu} \\ &=^\ddagger \text{RHom}_A(\text{id}_P \otimes_A^L \rho, \text{id}_M) \circ \text{RHom}_A(\text{ru}, \text{id}_M) \end{aligned}$$

of morphisms

$$\begin{aligned} G(M) &= \text{RHom}_A(P, M) \rightarrow \\ G(G(M)) &= \text{RHom}_A(P, \text{RHom}_A(P, M)) \cong \text{RHom}_A(P \otimes_A^L P, M) \end{aligned}$$

in  $\mathbf{D}(A^{\text{en}})$ . Because both  $\rho \otimes_A^L \text{id}_P$  and  $\text{id}_P \otimes_A^L \rho$  are isomorphisms in  $\mathbf{D}(A^{\text{en}})$ , it follows that the morphisms  $G(\tau_M)$  and  $\tau_{G(M)}$  are isomorphisms in  $\mathbf{D}(A \otimes B^{\text{op}})$ .  $\square$

**Lemma 6.13.** *Consider the functors  $F$  and  $G$  on  $\mathbf{D}(A \otimes B^{\text{op}})$  from Definition 6.4. For any  $M, N \in \mathbf{D}(A \otimes B^{\text{op}})$  there is a bijection*

$$\text{Hom}_{\mathbf{D}(A \otimes B^{\text{op}})}(F(M), N) \cong \text{Hom}_{\mathbf{D}(A \otimes B^{\text{op}})}(M, G(N)),$$

and it is functorial in  $M$  and  $N$ .

*Proof.* Choose a K-injective resolution  $N \rightarrow J$ , and a K-flat resolution  $\tilde{P} \rightarrow P$ , both in  $\mathbf{C}(A \otimes B^{\text{op}})$ . The usual Hom-tensor adjunction gives rise to an isomorphism

$$(6.14) \quad \text{Hom}_{A \otimes B^{\text{op}}}(\tilde{P} \otimes_A M, J) \cong \text{Hom}_{A \otimes B^{\text{op}}}(M, \text{Hom}_A(\tilde{P}, J))$$

in  $\mathbf{C}(\mathbb{K})$ . From this we deduce that  $\text{Hom}_A(\tilde{P}, J)$  is K-injective in  $\mathbf{C}(A \otimes B^{\text{op}})$ . We see that the isomorphism (6.14) represents an isomorphism

$$(6.15) \quad \text{RHom}_{A \otimes B^{\text{op}}}(P \otimes_A^L M, N) \cong \text{RHom}_{A \otimes B^{\text{op}}}(M, \text{RHom}_A(P, N))$$

in  $\mathbf{D}(\mathbb{K})$ . Taking  $H^0$  in (6.15) gives us the isomorphism

$$\text{Hom}_{\mathbf{D}(A \otimes B^{\text{op}})}(P \otimes_A^L M, N) \cong \text{Hom}_{\mathbf{D}(A \otimes B^{\text{op}})}(M, \text{RHom}_A(P, N))$$

in  $\mathbf{M}(\mathbb{K})$ . This is what we want.  $\square$

**Lemma 6.16.** *Consider the functors  $F$  and  $G$  on  $\mathbf{D}(A \otimes B^{\text{op}})$  from Definition 6.4. The kernel of  $F$  equals the kernel of  $G$ . Namely for any  $M \in \mathbf{D}(A \otimes B^{\text{op}})$  we have  $F(M) = 0$  if and only if  $G(M) = 0$ .*

*Proof.* We shall use the adjunction formula from Lemma 6.13, with  $N = M$ .

First assume  $F(M) = 0$ . Then  $\text{Hom}_{\mathbf{D}(A \otimes B^{\text{op}})}(F(M), M)$  is zero, and by Lemma 6.13 we see that  $\text{Hom}_{\mathbf{D}(A \otimes B^{\text{op}})}(M, G(M))$  is zero too. This implies that the morphism  $\tau_M : M \rightarrow G(M)$  is zero. Applying  $G$  to it we deduce that the morphism

$$G(\tau_M) : G(M) \rightarrow G(G(M))$$

is zero. But by Lemma 6.8(1) the pointed functor  $(G, \tau)$  is idempotent, and this means that  $G(\tau_M)$  is an isomorphism. Therefore  $G(M) = 0$ .

Now assume that  $G(M) = 0$ . Again using Lemma 6.13, but now in the reverse direction, we see that the morphism  $\sigma_M : F(M) \rightarrow M$  is zero. Therefore the morphism

$$F(\sigma_M) : F(F(M)) \rightarrow F(M)$$

is zero. But by Lemma 6.8(1) the copointed functor  $(F, \sigma)$  is idempotent, and this means that  $F(\sigma_M)$  is an isomorphism. Therefore  $F(M) = 0$ .  $\square$

Recall that Conventions 5.1 and 6.1 are in force.

**Theorem 6.17** (Abstract Equivalence). *Let  $A$  and  $B$  be flat rings, and let  $(P, \rho)$  be an idempotent copointed object in  $\mathbf{D}(A^{\text{en}})$ . Consider the triangulated functors*

$$F, G : \mathbf{D}(A \otimes B^{\text{op}}) \rightarrow \mathbf{D}(A \otimes B^{\text{op}})$$

*and the categories  $\mathbf{D}(A \otimes B^{\text{op}})_F$  and  $\mathbf{D}(A \otimes B^{\text{op}})_G$  from Definitions 6.4 and 6.6. The following hold:*

- (1) *The functor  $G$  is a right adjoint to  $F$ .*
- (2) *The copointed triangulated functor  $(F, \sigma)$  and the pointed triangulated functor  $(G, \tau)$  are idempotent.*
- (3) *The categories  $\mathbf{D}(A \otimes B^{\text{op}})_F$  and  $\mathbf{D}(A \otimes B^{\text{op}})_G$  are the essential images of the functors  $F$  and  $G$  respectively.*
- (4) *The functor*

$$F : \mathbf{D}(A \otimes B^{\text{op}})_G \rightarrow \mathbf{D}(A \otimes B^{\text{op}})_F$$

*is an equivalence of triangulated categories, with quasi-inverse  $G$ .*

*Proof.* (1) This is Lemma 6.13.

(2) This is Lemma 6.8.

(3) Take any  $M \in \mathbf{D}(A \otimes B^{\text{op}})_G$ . Then  $M \cong G(M)$ , so that  $M$  is in the essential image of  $G$ . Conversely, suppose there is an isomorphism  $\phi : M \xrightarrow{\sim} G(N)$  for some  $N \in \mathbf{D}(A \otimes B^{\text{op}})$ . We have to prove that  $\tau_M$  is an isomorphism. There is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi} & G(N) \\ \tau_M \downarrow & & \downarrow \tau_{G(N)} \\ G(M) & \xrightarrow{G(\phi)} & G(G(N)) \end{array}$$

in  $\mathbf{D}(A \otimes B^{\text{op}})$  with horizontal isomorphisms. By Lemma 6.8(1) the morphism  $\tau_{G(N)}$  is an isomorphism. Therefore  $\tau_M$  is an isomorphism.

A similar argument (with reversed arrows) tells us that the essential image of  $F$  is  $\mathbf{D}(A \otimes B^{\text{op}})_F$ .

(4) The morphism  $\rho : P \rightarrow A$  sits inside a distinguished triangle

$$(6.18) \quad P \xrightarrow{\rho} A \rightarrow N \xrightarrow{\Delta}$$

in  $D(A^{\text{en}})$ . Let us apply the functor  $P \otimes_A^L -$  to (6.18). We get a distinguished triangle

$$P \otimes_A^L P \xrightarrow{\text{ru} \circ (\text{id} \otimes_A^L \rho)} P \rightarrow P \otimes_A^L N \xrightarrow{\Delta}$$

in  $D(A^{\text{en}})$ . By the idempotence condition, the first morphism above is an isomorphism; and hence  $P \otimes_A^L N = 0$ . Therefore for any  $M \in D(A \otimes B^{\text{op}})$  the complex

$$F(N \otimes_A^L M) = P \otimes_A^L N \otimes_A^L M$$

is zero. Lemma 6.16 tells us that

$$G(N \otimes_A^L M) = \text{RHom}_A(P, N \otimes_A^L M)$$

is zero.

Now we go back to the distinguished triangle (6.18) and we apply to it the functor  $-\otimes_A^L M$ , and then the functor  $\text{RHom}_A(P, -)$ . The result is the distinguished triangle

$$\text{RHom}_A(P, P \otimes_A^L M) \xrightarrow{\alpha_M} \text{RHom}_A(P, M) \rightarrow \text{RHom}_A(P, N \otimes_A^L M) \xrightarrow{\Delta}$$

in  $D(A \otimes B^{\text{op}})$ . Because the third term is zero, it follows that  $\alpha_M : G(F(M)) \rightarrow G(M)$  is an isomorphism. If moreover  $M \in D(A \otimes B^{\text{op}})_G$ , then  $\tau_M$  is an isomorphism too, and thus we have an isomorphism

$$\tau_M^{-1} \circ \alpha_M : G(F(M)) \rightarrow M$$

that's functorial in  $M$ .

Similarly, if we apply the functor  $-\otimes_A^L P$  to (6.18), we get a distinguished triangle

$$P \otimes_A^L P \xrightarrow{\text{lu} \circ (\rho \otimes_A^L \text{id})} P \rightarrow N \otimes_A^L P \xrightarrow{\Delta}$$

in  $D(A^{\text{en}})$ . By the idempotence condition, the first morphism above is an isomorphism; and hence  $N \otimes_A^L P = 0$ . Therefore for any  $M \in D(A \otimes B^{\text{op}})$  the complex

$$G(\text{RHom}_A(N, M)) = \text{RHom}_A(P, \text{RHom}_A(N, M)) \cong \text{RHom}_A(N \otimes_A^L P, M)$$

is zero. Lemma 6.16 tells us that

$$F(\text{RHom}_A(N, M)) = P \otimes_A^L \text{RHom}_A(N, M)$$

is zero.

Next we apply the functor  $\text{RHom}_A(-, M)$ , and then the functor  $P \otimes_A^L -$ , to the distinguished triangle (6.18). We obtain distinguished triangle

$$P \otimes_A^L \text{RHom}_A(N, M) \rightarrow P \otimes_A^L M \xrightarrow{\beta_M} P \otimes_A^L \text{RHom}_A(P, M) \xrightarrow{\Delta}$$

in  $D(A \otimes B^{\text{op}})$ . By the previous calculation the first term in this triangle is zero, and so  $\beta_M : F(M) \rightarrow F(G(M))$  is an isomorphism. If moreover  $M \in D(A \otimes B^{\text{op}})_F$ , then  $\sigma_M$  is an isomorphism too, and thus we have an isomorphism

$$\beta_M \circ \sigma_M^{-1} : M \rightarrow F(G(M))$$

that's functorial in  $M$ . □

**Remark 6.19.** The content of this section is not very hard to extend to the DG setup:  $A$  and  $B$  can be nonpositive K-flat central DG  $\mathbb{K}$ -rings, as in Remark 5.22.



**Remark 6.20.** Theorem 6.17 can be interpreted as saying that every idempotent copointed object in  $D(A^{\text{en}})$  induces a recollement on the category  $D(A \otimes B^{\text{op}})$ . This is not unexpected; cf. [Kr, Paragraph 4.13.1]. This perspective, in the context of weakly stable torsion classes, will be examined in a future paper.

## 7. FROM TORSION CLASSES TO COPOINTED OBJECTS

In this section we prove Theorems 7.12 and 7.18. Conventions 5.1 and 6.1 are in place. Recall that  $\mathbb{K}$  is a commutative base ring,  $A$  and  $B$  are flat central  $\mathbb{K}$ -rings. We write  $\otimes$  as shorthand for  $\otimes_{\mathbb{K}}$ . There is a restriction functor

$$\text{Rest} : M(A \otimes B^{\text{op}}) \rightarrow M(A)$$

that forgets the right  $B$ -module structure. This extends to restriction functors on  $C(-)$  and  $D(-)$  with the same notation.

Let  $\mathsf{T}$  be a torsion class in  $M(A)$ . The torsion functor associated to  $\mathsf{T}$  is  $\Gamma_{\mathsf{T}}$ , and the Gabriel filter is  $\text{Filt}(\mathsf{T})$ . See Section 3 for a review of these concepts.

**Definition 7.1.** Given a torsion class  $\mathsf{T} \subseteq M(A)$ , the *bimodule torsion class*  $\mathsf{T}_{B^{\text{op}}} \subseteq M(A \otimes B^{\text{op}})$  is defined as follows:

$$\mathsf{T}_{B^{\text{op}}} := \{M \in M(A \otimes B^{\text{op}}) \mid \text{Rest}(M) \in \mathsf{T}\}.$$

The torsion functor  $\Gamma_{\mathsf{T}_{B^{\text{op}}}}$  on  $M(A \otimes B^{\text{op}})$  satisfies this formula: there is equality

$$(7.2) \quad \Gamma_{\mathsf{T}_{B^{\text{op}}}}(M) = \varinjlim_{\mathfrak{a} \in \text{Filt}(\mathsf{T})} \text{Hom}_A(A/\mathfrak{a}, M)$$

of submodules of  $M$ . There is a morphism of functors

$$(7.3) \quad \sigma : \Gamma_{\mathsf{T}_{B^{\text{op}}}} \rightarrow \text{Id}_{M(A \otimes B^{\text{op}})}.$$

The pair  $(\Gamma_{\mathsf{T}_{B^{\text{op}}}}, \sigma)$  is an idempotent copointed functor on the category  $M(A \otimes B^{\text{op}})$ . The copointed functor  $(\Gamma_{\mathsf{T}_{B^{\text{op}}}}, \sigma)$  extends in the obvious way to complexes, giving rise to an idempotent copointed functor on the category  $C(A \otimes B^{\text{op}})$ .

By definition the diagram of functors

$$(7.4) \quad \begin{array}{ccc} M(A \otimes B^{\text{op}}) & \xrightarrow{\Gamma_{\mathsf{T}_{B^{\text{op}}}}} & M(A \otimes B^{\text{op}}) \\ \text{Rest} \downarrow & & \downarrow \text{Rest} \\ M(A) & \xrightarrow{\Gamma_{\mathsf{T}}} & M(A) \end{array}$$

is commutative. Moreover, for any  $M \in M(A \otimes B^{\text{op}})$  there is equality

$$(7.5) \quad \text{Rest}(\sigma_M) = \sigma_{\text{Rest}(M)}$$

of homomorphisms

$$\text{Rest}(\Gamma_{\mathsf{T}_{B^{\text{op}}}}(M)) \rightarrow \text{Rest}(M)$$

in  $M(A)$ , because they are both the inclusion of the  $\mathbb{K}$ -module  $\Gamma_{\mathsf{T}}(M)$  into the  $\mathbb{K}$ -module  $M$ .

In case  $B = \mathbb{K}$ , so that  $A \otimes B^{\text{op}} = A$  and  $\mathsf{T}_{B^{\text{op}}} = \mathsf{T}$ , we are back in the situation studied in Section 3.

Here is a definition resembling Definition 5.7.

**Definition 7.6.** A complex  $I \in C(A \otimes B^{\text{op}})$  is called  *$\mathsf{T}$ -flasque over  $A$*  if  $\text{Rest}(I) \in C(A)$  is  $\mathsf{T}$ -flasque, in the sense of Definition 3.5(2).

The next two lemmas can be deduced from standard properties of right derived functors. Still, we shall go through the details, because they will serve as references in subsequent constructions.

**Lemma 7.7.** *The functor  $\Gamma_{\mathcal{T}_{B^{\text{op}}}}$  has a right derived functor*

$$(\mathbf{R}\Gamma_{\mathcal{T}_{B^{\text{op}}}, \xi^{\mathbf{R}}) : \mathbf{D}(A \otimes B^{\text{op}}) \rightarrow \mathbf{D}(A \otimes B^{\text{op}}).$$

*If  $M \in \mathbf{D}(A \otimes B^{\text{op}})$  is a  $\mathcal{T}$ -flasque complex over  $A$ , then*

$$\xi_M^{\mathbf{R}} : \Gamma_{\mathcal{T}_{B^{\text{op}}}}(M) \rightarrow \mathbf{R}\Gamma_{\mathcal{T}_{B^{\text{op}}}}(M)$$

*is an isomorphism.*

*Proof.* For any  $M \in \mathbf{C}(A \otimes B^{\text{op}})$  we choose a K-injective resolution  $\zeta_M : M \rightarrow I_M$  in  $\mathbf{C}(A \otimes B^{\text{op}})$ . By Lemma 5.8 the object  $\text{Rest}(I_M) \in \mathbf{C}(A)$  is K-injective, and hence it is  $\mathcal{T}$ -flasque. We define  $\mathbf{R}\Gamma_{\mathcal{T}_{B^{\text{op}}}}(M) := \Gamma_{\mathcal{T}_{B^{\text{op}}}}(I_M)$  and

$$\xi_M^{\mathbf{R}} := \Gamma_{\mathcal{T}_{B^{\text{op}}}}(\zeta_M) : \Gamma_{\mathcal{T}_{B^{\text{op}}}}(M) \rightarrow \mathbf{R}\Gamma_{\mathcal{T}_{B^{\text{op}}}}(M).$$

If  $M$  is a  $\mathcal{T}$ -flasque complex over  $A$ , then the homomorphism

$$\Gamma_{\mathcal{T}_{B^{\text{op}}}}(\zeta_M) : \Gamma_{\mathcal{T}_{B^{\text{op}}}}(M) \rightarrow \Gamma_{\mathcal{T}_{B^{\text{op}}}}(I_M)$$

in  $\mathbf{C}(A \otimes B)$  is a quasi-isomorphism, and hence  $\xi_M^{\mathbf{R}}$  is an isomorphism.  $\square$

Here is the bimodule version of Lemma 2.4.

**Lemma 7.8.** *Consider the triangulated functor  $(\mathbf{R}\Gamma_{\mathcal{T}_{B^{\text{op}}}, \xi^{\mathbf{R}})$  from Lemma 7.7. There is a unique morphism*

$$\sigma^{\mathbf{R}} : \mathbf{R}\Gamma_{\mathcal{T}_{B^{\text{op}}}} \rightarrow \text{Id}$$

*of triangulated functors from  $\mathbf{D}(A \otimes B^{\text{op}})$  to itself, satisfying this condition: for any  $M \in \mathbf{D}(A \otimes B^{\text{op}})$  the diagram*

$$\begin{array}{ccc} \Gamma_{\mathcal{T}_{B^{\text{op}}}}(M) & \xrightarrow{\xi_M^{\mathbf{R}}} & \mathbf{R}\Gamma_{\mathcal{T}_{B^{\text{op}}}}(M) \\ & \searrow \sigma_M & \downarrow \sigma_M^{\mathbf{R}} \\ & & M \end{array}$$

*in  $\mathbf{D}(A \otimes B^{\text{op}})$  is commutative.*

*Proof.* When  $I$  is  $\mathcal{T}$ -flasque over  $A$  the morphism  $\xi_I^{\mathbf{R}}$  is an isomorphism, and so the morphism  $\sigma_I^{\mathbf{R}} : \mathbf{R}\Gamma_{\mathcal{T}_{B^{\text{op}}}}(I) \rightarrow I$  must be  $\sigma_I^{\mathbf{R}} = \sigma_I \circ (\xi_I^{\mathbf{R}})^{-1}$ . This implies the uniqueness of the morphism of functors  $\sigma^{\mathbf{R}}$ .

For existence we use the chosen K-injective resolutions  $\zeta_M : M \rightarrow I_M$  from the proof of Lemma 7.7, and define

$$\sigma_{I_M}^{\mathbf{R}} := \sigma_{I_M} \circ (\xi_{I_M}^{\mathbf{R}})^{-1}$$

and

$$\sigma_M^{\mathbf{R}} := \zeta_M^{-1} \circ \sigma_{I_M}^{\mathbf{R}} \circ \mathbf{R}\Gamma_{\mathcal{T}_{B^{\text{op}}}}(\zeta_M).$$

$\square$

We now have a triangulated copointed functor  $(\mathbf{R}\Gamma_{\mathcal{T}_{B^{\text{op}}}, \sigma^{\mathbf{R}})$  on the category  $\mathbf{D}(A \otimes B^{\text{op}})$ .

**Lemma 7.9.** *Consider the triangulated copointed functors  $(R\Gamma_{\mathbf{T}}, \sigma^R)$  and  $(R\Gamma_{\mathbf{T}_{B^{\text{op}}}}, \sigma^R)$  on the categories  $D(A)$  and  $D(A \otimes B^{\text{op}})$  respectively. There is an isomorphism*

$$\eta : \text{Rest} \circ R\Gamma_{\mathbf{T}_{B^{\text{op}}}} \xrightarrow{\cong} R\Gamma_{\mathbf{T}} \circ \text{Rest}$$

*of triangulated functors  $D(A \otimes B^{\text{op}}) \rightarrow D(A)$ , such that for any object  $M \in D(A \otimes B^{\text{op}})$  there is equality*

$$(7.10) \quad \text{Rest}(\sigma_M^R) = \sigma_{\text{Rest}(M)}^R \circ \eta_M$$

*of morphisms  $\text{Rest}(R\Gamma_{\mathbf{T}_{B^{\text{op}}}}(M)) \rightarrow \text{Rest}(M)$  in  $D(A)$ .*

*Proof.* We use the K-injective resolutions  $\zeta_M : M \rightarrow I_M$  in  $C(A \otimes B^{\text{op}})$  from the proof of Lemma 7.7 to present the functor  $R\Gamma_{\mathbf{T}_{B^{\text{op}}}}$ . Let us also choose a system of K-injective resolutions  $\theta_N : N \rightarrow J_N$  in the category  $C(A)$ , and use that to present the functor  $R\Gamma_{\mathbf{T}}$ .

For any complex  $M \in C(A \otimes B^{\text{op}})$ , with image  $N := \text{Rest}(M) \in C(A)$ , there exists a quasi-isomorphism  $\delta_M : \text{Rest}(I_M) \rightarrow J_N$  in  $C(A)$ , unique up to homotopy, such that the diagram

$$\begin{array}{ccc} \text{Rest}(M) & \xrightarrow{\text{Rest}(\zeta_M)} & \text{Rest}(I_M) \\ \text{id} \downarrow & & \downarrow \delta_M \\ N & \xrightarrow{\theta_N} & J_N \end{array}$$

is commutative up to homotopy. Because both  $\text{Rest}(I_M)$  and  $J_N$  are  $\mathbf{T}$ -flasque over  $A$ , the homomorphism

$$\eta_M := \Gamma_{\mathbf{T}}(\delta_M) : \Gamma_{\mathbf{T}}(\text{Rest}(I_M)) \rightarrow \Gamma_{\mathbf{T}}(J_N)$$

in  $C(A)$  is a quasi-isomorphism. The commutativity of diagram (7.4) implies that

$$\Gamma_{\mathbf{T}}(\text{Rest}(I_M)) = \text{Rest}(\Gamma_{\mathbf{T}_{B^{\text{op}}}}(I_M))$$

as submodules of  $\text{Rest}(I_M)$ . Thus, as  $M$  varies, we obtain an isomorphism of triangulated functors

$$\eta : \text{Rest} \circ R\Gamma_{\mathbf{T}_{B^{\text{op}}}} \xrightarrow{\cong} R\Gamma_{\mathbf{T}} \circ \text{Rest}.$$

Finally, formula (7.5) says that  $\text{Rest}(\sigma_{I_M}^R) = \sigma_{\text{Rest}(I_M)}^R$  as homomorphisms

$$\text{Rest}(\Gamma_{\mathbf{T}_{B^{\text{op}}}}(I_M)) \rightarrow \text{Rest}(I_M)$$

in  $C(A)$ . The construction of  $\sigma_M^R$  in the proof of Lemma 7.8 shows that equality (7.10) holds.  $\square$

In view of Lemma 7.9, there is no harm to stop using the longhand notation  $\mathbf{T}_{B^{\text{op}}}$  for the bimodule torsion class in  $M(A \otimes B^{\text{op}})$ . From here on we shall mostly use the notation  $\mathbf{T}$  to denote both the original torsion class and the bimodule torsion class. Thus we shall usually write  $R\Gamma_{\mathbf{T}}$  for the derived torsion functors on  $D(A)$  and on  $D(A \otimes B^{\text{op}})$ , and the context will determine which category is involved. This simplified notation will be especially helpful because the ring  $B$  will be varying.

For the next definition we take  $B = A$ , so that  $A \otimes B^{\text{op}} = A^{\text{en}}$  and, in the longhand notation, the bimodule torsion class is  $\mathbf{T}_{A^{\text{op}}} \subseteq M(A^{\text{en}})$ .

The category  $D(A^{\text{en}})$  carries a monoidal structure – see Proposition 5.17. Copointed objects in monoidal categories were introduced in Definition 6.2.

**Definition 7.11.** Consider the copointed triangulated functor  $(R\Gamma_{\mathbf{T}}, \sigma^{\mathbf{R}})$  on  $D(A^{\text{en}})$ . Define the object  $P := R\Gamma_{\mathbf{T}}(A) \in D(A^{\text{en}})$  and the morphism

$$\rho : P \rightarrow A, \quad \rho := \sigma_A^{\mathbf{R}}$$

in  $D(A^{\text{en}})$ . We call the pair  $(P, \rho)$  the *copointed object in  $D(A^{\text{en}})$  induced by  $\mathbf{T}$* .

Recall that Conventions 5.1 and 6.1 are assumed. In the next theorem,  $B$  is an arbitrary (flat central)  $\mathbb{K}$ -ring; it could, for instance, be  $A$  or  $\mathbb{K}$ . By Proposition 5.17 there is a left monoidal action  $- \otimes_A^{\mathbf{L}} -$  of  $D(A^{\text{en}})$  on  $D(A \otimes B^{\text{op}})$ .

**Theorem 7.12** (Representability of Derived Torsion). *Let  $A$  and  $B$  be flat rings. Let  $\mathbf{T}$  be a quasi-compact, finite dimensional, weakly stable torsion class in  $\mathbf{M}(A)$ , and let  $(P, \rho)$  be the copointed object in  $D(A^{\text{en}})$  from Definition 7.11. There is an isomorphism*

$$\gamma : P \otimes_A^{\mathbf{L}} - \xrightarrow{\sim} R\Gamma_{\mathbf{T}}$$

*of triangulated functors from  $D(A \otimes B^{\text{op}})$  to itself, such that for any complex  $M$  the diagram*

$$\begin{array}{ccc} P \otimes_A^{\mathbf{L}} M & \xrightarrow{\gamma_M} & R\Gamma_{\mathbf{T}}(M) \\ \rho \otimes_A^{\mathbf{L}} \text{id} \downarrow & & \downarrow \sigma_M^{\mathbf{R}} \\ A \otimes_A^{\mathbf{L}} M & \xrightarrow{\text{lu}} & M \end{array}$$

*in  $D(A \otimes B^{\text{op}})$  is commutative.*

*Proof.* We begin by constructing the morphism of triangulated functors  $\gamma$ . For any complex  $M \in \mathbf{C}(A \otimes B^{\text{op}})$  we choose a K-injective resolution  $\zeta_M : M \rightarrow I_M$ , and a K-flat resolution  $\theta_M : Q_M \rightarrow M$ , both in  $\mathbf{C}(A \otimes B^{\text{op}})$ . Note that  $I_M$  is  $\mathbf{T}$ -flasque over  $A$ , and  $Q_M$  is K-flat over  $A$ . We use these choices for presentations of the right derived functor

$$(R\Gamma_{\mathbf{T}}, \xi^{\mathbf{R}}) : D(A \otimes B^{\text{op}}) \rightarrow D(A \otimes B^{\text{op}})$$

and the left derived bifunctor

$$(- \otimes_A^{\mathbf{L}} -, \xi^{\mathbf{L}}) : D(A^{\text{en}}) \times D(A \otimes B) \rightarrow D(A \otimes B^{\text{op}}).$$

Let us also choose a K-injective resolution  $\epsilon : A \rightarrow J$  in  $\mathbf{C}(A^{\text{en}})$ . With these choices we have the following presentations:  $P = \Gamma_{\mathbf{T}}(J)$ ,  $R\Gamma_{\mathbf{T}}(M) = \Gamma_{\mathbf{T}}(I_M)$  and  $P \otimes_A^{\mathbf{L}} M = \Gamma_{\mathbf{T}}(J) \otimes_A Q_M$ .

Let  $N \in \mathbf{C}(A \otimes B^{\text{op}})$ . Given homogeneous elements  $x \in \Gamma_{\mathbf{T}}(J)$  and  $n \in N$ , the tensor  $x \otimes n$  belongs to  $\Gamma_{\mathbf{T}}(J \otimes_A N)$ ; cf. formula (7.2). In this way we obtain a homomorphism

$$(7.13) \quad \tilde{\gamma}_N : \Gamma_{\mathbf{T}}(J) \otimes_A N \rightarrow \Gamma_{\mathbf{T}}(J \otimes_A N)$$

in  $\mathbf{C}(A \otimes B^{\text{op}})$ , which is functorial in  $N$ .

Consider a complex  $M \in \mathbf{C}(A \otimes B^{\text{op}})$ . We have the solid diagram

$$(7.14) \quad \begin{array}{ccccc} A \otimes_A Q_M & \xrightarrow{\text{lu}} & Q_M & \xrightarrow{\theta_M} & M \\ \epsilon \otimes_A \text{id} \downarrow & & & & \downarrow \zeta_M \\ J \otimes_A Q_M & \dashrightarrow & \dashrightarrow & \xrightarrow{\chi_M} & \dashrightarrow I_M \end{array}$$

in  $\mathbf{C}(A \otimes B^{\text{op}})$ . The homomorphisms  $\epsilon \otimes_A \text{id}$ ,  $\theta_M \circ \text{lu}$  and  $\zeta_M$  in  $\mathbf{C}(A \otimes B^{\text{op}})$  are quasi-isomorphisms. Because  $I_M$  is K-injective in  $\mathbf{C}(A \otimes B^{\text{op}})$ , there is a quasi-isomorphism

$$(7.15) \quad \chi_M : J \otimes_A Q_M \rightarrow I_M$$

that makes this diagram commutative up to homotopy.

We now form the following diagram

$$(7.16) \quad \begin{array}{ccccc} \Gamma_{\mathbf{T}}(J) \otimes_A Q_M & \xrightarrow{\tilde{\gamma}_{Q_M}} & \Gamma_{\mathbf{T}}(J \otimes_A Q_M) & \xrightarrow{\Gamma_{\mathbf{T}}(\chi_M)} & \Gamma_{\mathbf{T}}(I_M) \\ \sigma_J \otimes_A \text{id} \downarrow & & \sigma_{J \otimes_A Q_M} \downarrow & & \sigma_{I_M} \downarrow \\ J \otimes_A Q_M & \xrightarrow{\text{id}} & J \otimes_A Q_M & \xrightarrow{\chi_M} & I_M \\ \epsilon \otimes_A \text{id} \uparrow & & \epsilon \otimes_A \text{id} \uparrow & & \zeta_M \uparrow \\ A \otimes_A Q_M & \xrightarrow{\text{id}} & A \otimes_A Q_M & \xrightarrow{\theta_M \circ \text{lu}} & M \end{array}$$

in  $\mathbf{C}(A \otimes B^{\text{op}})$ . Here  $\tilde{\gamma}_{Q_M}$  is the homomorphism from (7.13). The diagram (7.16) is commutative up to homotopy. (Actually all small squares, except the bottom right one, are commutative in the strict sense.) The vertical arrows  $\epsilon \otimes_A \text{id}$  and  $\zeta_M$  are quasi-isomorphisms. Passing to  $\mathbf{D}(A \otimes B^{\text{op}})$  we get a commutative diagram, with vertical isomorphisms between the second and third rows. The diagram with the four extreme objects only is the one we are looking for. By construction it is a commutative diagram in  $\mathbf{D}(A \otimes B^{\text{op}})$ , and it is functorial in  $M$ . The morphism

$$\gamma_M : P \otimes_A^{\mathbf{L}} M \rightarrow \mathbf{R}\Gamma_{\mathbf{T}}(M)$$

is represented by  $\Gamma_{\mathbf{T}}(\chi_M) \circ \tilde{\gamma}_{Q_M}$ .

It remains to prove that  $\gamma_M$  is an isomorphism for any  $M \in \mathbf{D}(A \otimes B^{\text{op}})$ . Because the functor  $\text{Rest}$  is conservative, it suffices to prove that the morphism

$$\text{Rest}(\gamma_M) : \text{Rest}(P \otimes_A^{\mathbf{L}} M) \rightarrow \text{Rest}(\mathbf{R}\Gamma_{\mathbf{T}}(M))$$

in  $\mathbf{D}(A)$  is an isomorphism. Going over all the details of the construction above, and noting that  $\zeta_M : M \rightarrow I_M$  and  $\theta_M : Q_M \rightarrow M$  are K-flat and K-injective resolutions, respectively, also in  $\mathbf{C}(A)$ , we might as well forget about the ring  $B$ .

Another way to justify the elimination of  $B$  is this: we want to prove that the homomorphism

$$\Gamma_{\mathbf{T}}(\chi_M) \circ \tilde{\gamma}_{Q_M} : \Gamma_{\mathbf{T}}(J) \otimes_A Q_M \rightarrow \Gamma_{\mathbf{T}}(I)$$

in  $\mathbf{C}(A \otimes B^{\text{op}})$  is a quasi-isomorphism. For that we can forget about the  $B$ -module structure.

So now we are in the case  $B = \mathbb{K}$ ,  $A \otimes B^{\text{op}} = A$  and (in longhand)  $\mathbf{T}_{B^{\text{op}}} = \mathbf{T}$ , and we want to prove that  $\gamma_M$  is an isomorphism for any  $M \in \mathbf{D}(A)$ . By Theorem 3.8 the functor  $\mathbf{R}\Gamma_{\mathbf{T}}$  on  $\mathbf{D}(A)$  is quasi-compact. The functor  $P \otimes_A^{\mathbf{L}} -$  is also quasi-compact. This means that we can use [PSY2, Lemma 4.1], and it tells us that it suffices to prove that  $\gamma_M$  is an isomorphism for  $M = A$ .

Let us examine the morphism  $\gamma_A$ , i.e.  $\gamma_M$  for  $M = A$ . We can choose the K-flat resolution  $\theta_A : Q_A \rightarrow A$  in  $\mathbf{C}(A)$  to be the identity of  $A$ . Also, we can choose the K-injective resolution  $\zeta_A : A \rightarrow I_A$  in  $\mathbf{C}(A)$  to be the restriction of  $\epsilon : A \rightarrow J$ . Then

the homomorphism  $\chi_A : J \otimes_A Q_A \rightarrow I_A$  in diagram (7.14) can be chosen to be  $\chi_A = \text{id} \otimes_A \text{id}$ . We get a commutative diagram

$$\begin{array}{ccccc}
 \Gamma_{\mathbb{T}}(J) \otimes_A Q_A & \xrightarrow{\tilde{\gamma}_{Q_A}} & \Gamma_{\mathbb{T}}(J \otimes_A Q_A) & \xrightarrow{\Gamma_{\mathbb{T}}(\chi_A)} & \Gamma_{\mathbb{T}}(I_A) \\
 \text{id} \otimes_A \text{id} \downarrow & & \Gamma_{\mathbb{T}}(\text{id} \otimes_A \text{id}) \downarrow & & \text{id} \downarrow \\
 \Gamma_{\mathbb{T}}(J) \otimes_A A & \xrightarrow{\tilde{\gamma}_A} & \Gamma_{\mathbb{T}}(J \otimes_A A) & \xrightarrow{\Gamma_{\mathbb{T}}(\text{ru})} & \Gamma_{\mathbb{T}}(J)
 \end{array}$$

in  $\mathcal{C}(A)$ . The horizontal arrows in the second row, and the vertical arrows, are all bijective. We conclude that  $\Gamma_{\mathbb{T}}(\chi_A) \circ \tilde{\gamma}_{Q_A}$  is bijective. Hence  $\gamma_A$  is an isomorphism in  $\mathcal{D}(A)$ .  $\square$

**Remark 7.17.** Theorem 7.12 resembles [WZ, Lemma 3.4]. There the authors only considered the case where  $A$  is a complete semilocal noetherian ring (central over a base field  $\mathbb{K}$ ), and  $\mathbb{T}$  is torsion at the Jacobson radical of  $A$ .

Again we remind that Conventions 5.1 and 6.1 are in place. The copointed object induced by a torsion class was introduced in Definition 7.11.

**Theorem 7.18.** *Let  $A$  be a flat ring, and let  $\mathbb{T}$  be a quasi-compact, finite dimensional, weakly stable torsion class in  $\mathcal{M}(A)$ . Let  $(P, \rho)$  be the copointed object in the monoidal category  $\mathcal{D}(A^{\text{en}})$  that is induced by  $\mathbb{T}$ . Then  $(P, \rho)$  is an idempotent copointed object.*

*Proof.* We shall start by proving that

$$(7.19) \quad \text{lu} \circ (\rho \otimes_A^L \text{id}) : P \otimes_A^L P \rightarrow P$$

is an isomorphism in  $\mathcal{D}(A^{\text{en}})$ . Because the forgetful functor  $\text{Rest} : \mathcal{D}(A^{\text{en}}) \rightarrow \mathcal{D}(A)$  is conservative, it is enough if we prove that  $\text{Rest}(\text{lu} \circ (\rho \otimes_A^L \text{id}))$  is an isomorphism. Let us introduce the temporary notation  $P' := \text{Rest}(P) \in \mathcal{D}(A)$ . With this notation, what we have to show is that

$$(7.20) \quad \text{lu} \circ (\rho \otimes_A^L \text{id}) : P \otimes_A^L P' \rightarrow P'$$

is an isomorphism in  $\mathcal{D}(A)$ .

Consider Theorem 7.12 with  $B = \mathbb{K}$  and  $M = P' \in \mathcal{D}(A)$ . There is a commutative diagram

$$\begin{array}{ccc}
 P \otimes_A^L P' & \xrightarrow{\gamma_{P'}} & \text{R}\Gamma_{\mathbb{T}}(P') \\
 \rho \otimes_A^L \text{id} \downarrow & & \downarrow \sigma_{P'}^{\text{R}} \\
 A \otimes_A^L P' & \xrightarrow{\text{lu}} & P'
 \end{array}$$

in  $\mathcal{D}(A)$ , and the horizontal arrows are isomorphisms. It suffices to prove that  $\sigma_{P'}^{\text{R}} : \text{R}\Gamma_{\mathbb{T}}(P') \rightarrow P'$  is an isomorphism in  $\mathcal{D}(A)$ . But by Lemma 7.9 there is an isomorphism  $P' \cong \text{R}\Gamma_{\mathbb{T}}(A')$ , where  $A' := \text{Rest}(A) \in \mathcal{D}(A)$ . So what we need to prove is that

$$\sigma_{\text{R}\Gamma_{\mathbb{T}}(A')}^{\text{R}} : \text{R}\Gamma_{\mathbb{T}}(\text{R}\Gamma_{\mathbb{T}}(A')) \rightarrow \text{R}\Gamma_{\mathbb{T}}(A')$$

is an isomorphism in  $\mathcal{D}(A)$ . This is true because the copointed triangulated functor  $(\text{R}\Gamma_{\mathbb{T}}, \sigma^{\text{R}})$  on  $\mathcal{D}(A)$  is idempotent; see Theorem 3.8.

Now we are going to prove that

$$(7.21) \quad \text{ru} \circ (\text{id} \otimes_A^L \rho) : P \otimes_A^L P \rightarrow P$$

is an isomorphism in  $D(A^{\text{en}})$ .

We have this commutative diagram in  $D(A^{\text{en}})$  :

$$(7.22) \quad \begin{array}{ccc} P \otimes_A^L P & \xleftarrow{\text{id} \otimes_A^L \text{ru}} & P \otimes_A^L P \otimes_A^L A \\ \text{id} \otimes_A^L \rho \downarrow & & \downarrow \text{id} \otimes_A^L \rho \otimes_A^L \text{id} \\ P \otimes_A^L A & \xleftarrow{\text{id} \otimes_A^L \text{ru}} & P \otimes_A^L A \otimes_A^L A \end{array}$$

Using Theorem 7.12 with  $B = A$  and  $M = A \in D(A^{\text{en}})$ , we have this commutative diagram in  $D(A^{\text{en}})$  :

$$(7.23) \quad \begin{array}{ccc} P \otimes_A^L A & \xrightarrow{\gamma_A} & \text{R}\Gamma_{\mathbf{T}}(A) \\ \rho \otimes_A^L \text{id} \downarrow & & \downarrow \sigma_A^{\text{R}} \\ A \otimes_A^L A & \xrightarrow{\text{lu}} & A \end{array}$$

Applying the functor  $P \otimes_A^L -$  to this diagram, we obtain this commutative diagram

$$(7.24) \quad \begin{array}{ccc} P \otimes_A^L P \otimes_A^L A & \xrightarrow{\text{id} \otimes_A^L \gamma_A} & P \otimes_A^L \text{R}\Gamma_{\mathbf{T}}(A) \\ \text{id} \otimes_A^L \rho \otimes_A^L \text{id} \downarrow & & \downarrow \text{id} \otimes_A^L \sigma_A^{\text{R}} \\ P \otimes_A^L A \otimes_A^L A & \xrightarrow{\text{id} \otimes_A^L \text{lu}} & P \otimes_A^L A \end{array}$$

in  $D(A^{\text{en}})$ . The last move is using the fact that  $\gamma$  is an isomorphism of functors; this yields the next commutative diagram:

$$(7.25) \quad \begin{array}{ccc} P \otimes_A^L \text{R}\Gamma_{\mathbf{T}}(A) & \xrightarrow{\gamma_{\text{R}\Gamma_{\mathbf{T}}(A)}} & \text{R}\Gamma_{\mathbf{T}}(\text{R}\Gamma_{\mathbf{T}}(A)) \\ \text{id} \otimes_A^L \sigma_A^{\text{R}} \downarrow & & \downarrow \text{R}\Gamma_{\mathbf{T}}(\sigma_A^{\text{R}}) \\ P \otimes_A^L A & \xrightarrow{\gamma_A} & \text{R}\Gamma_{\mathbf{T}}(A) \end{array}$$

All horizontal arrows in diagrams (7.22), (7.23), (7.24) and (7.25) are isomorphisms. Since  $\text{ru}$  is an isomorphism, to prove that (7.21) is an isomorphism, it is enough to prove that the morphism  $\text{id} \otimes_A^L \rho$  in (7.22) is an isomorphism. Passing horizontally from diagram (7.22) to diagram (7.25), we see that it is enough to prove that the morphism (written in longhand)

$$(7.26) \quad \text{R}\Gamma_{\mathbf{T}_{A^{\text{op}}}}(\sigma_A^{\text{R}}) : \text{R}\Gamma_{\mathbf{T}_{A^{\text{op}}}}(\text{R}\Gamma_{\mathbf{T}_{A^{\text{op}}}}(A)) \rightarrow \text{R}\Gamma_{\mathbf{T}_{A^{\text{op}}}}(A)$$

is an isomorphism in  $D(A^{\text{en}})$ . Because the functor  $\text{Rest}$  is conservative, and by Lemma 7.9, it suffices to prove that

$$\text{R}\Gamma_{\mathbf{T}}(\sigma_{A'}^{\text{R}}) : \text{R}\Gamma_{\mathbf{T}}(\text{R}\Gamma_{\mathbf{T}}(A')) \rightarrow \text{R}\Gamma_{\mathbf{T}}(A')$$

is an isomorphism in  $D(A)$ , where, as before, we write  $A' := \text{Rest}(A) \in D(A)$ . This is true because the copointed triangulated functor  $(\text{R}\Gamma_{\mathbf{T}}, \sigma^{\text{R}})$  on  $D(A)$  is idempotent; see Theorem 3.8.  $\square$

**Remark 7.27.** As explained in Remark 5.22, the flatness assumption can be circumvented using K-flat DG ring resolutions. However, the technicalities involved

in proving the nonflat versions of Theorems 7.18 and 7.12 turned out to be quite formidable. These nonflat generalizations will appear in a future paper.

## 8. NONCOMMUTATIVE MGM EQUIVALENCE AND SYMMETRIC DERIVED TORSION

In this final section we prove Theorems 8.3 and 8.9, that are Theorems 0.6 and 0.7 respectively in the Introduction. Conventions 5.1 and 6.1 are in place; in particular,  $\mathbb{K}$  is a commutative base ring, and  $A$  and  $B$  are flat central  $\mathbb{K}$ -rings.

Let us briefly recall torsion in bimodule categories, as developed in Section 7. Suppose we are given torsion classes  $\mathsf{T} \subseteq \mathsf{M}(A)$  and  $\mathsf{S}^{\text{op}} \subseteq \mathsf{M}(B^{\text{op}})$ . These lift, or extend, to torsion classes  $\mathsf{T}, \mathsf{S}^{\text{op}} \subseteq \mathsf{M}(A \otimes B^{\text{op}})$ . An  $A$ - $B$ -bimodule  $M$  is  $\mathsf{T}$ -torsion, by definition, if it is so as a left  $A$ -module (i.e. after forgetting the right  $B$ -module structure). Likewise for  $\mathsf{S}^{\text{op}}$ . In the longhand notation used in the beginning of Section 7, we would denote these extended torsion classes in  $\mathsf{M}(A \otimes B^{\text{op}})$  by  $\mathsf{T}_{B^{\text{op}}}$  and  $\mathsf{S}_A^{\text{op}}$ .

In Section 7 we already had the derived torsion functor

$$\mathsf{R}\Gamma_{\mathsf{T}} : \mathsf{D}(A \otimes B^{\text{op}}) \rightarrow \mathsf{D}(A \otimes B^{\text{op}}).$$

The same game can be played after relacing the pair of rings  $(A, B)$  with the pair

$$(B^{\text{op}}, (A^{\text{op}})^{\text{op}}) = (B^{\text{op}}, A).$$

This will give rise to a triangulated functor

$$\mathsf{R}\Gamma_{\mathsf{S}^{\text{op}}} : \mathsf{D}(A \otimes B^{\text{op}}) \rightarrow \mathsf{D}(A \otimes B^{\text{op}}).$$

For any  $M \in \mathsf{D}(A \otimes B^{\text{op}})$  and any  $q \in \mathbb{Z}$ , the cohomologies  $H^q(\mathsf{R}\Gamma_{\mathsf{T}}(M))$  and  $H^q(\mathsf{R}\Gamma_{\mathsf{S}^{\text{op}}}(M))$  are objects of  $\mathsf{M}(A \otimes B^{\text{op}})$ , so we can ask whether they are  $\mathsf{T}$ -torsion or  $\mathsf{S}^{\text{op}}$ -torsion.

**Example 8.1.** Suppose  $\mathfrak{a} \subseteq A$  and  $\mathfrak{b}^{\text{op}} \subseteq B^{\text{op}}$  are two-sided ideals that are finitely generated as left ideals. Note that  $\mathfrak{b}^{\text{op}}$  can be viewed as a two-sided ideal  $\mathfrak{b} \subseteq B$ , and then it is finitely generated as a *right ideal*.

In the notation of Definition 3.3, there are torsion classes  $\mathsf{T}_{\mathfrak{a}} \subseteq \mathsf{M}(A)$  and  $\mathsf{T}_{\mathfrak{b}^{\text{op}}} \subseteq \mathsf{M}(B^{\text{op}})$ . Let us define  $\mathsf{T} := \mathsf{T}_{\mathfrak{a}}$  and  $\mathsf{S}^{\text{op}} := \mathsf{T}_{\mathfrak{b}^{\text{op}}}$ . This places us in the situation described above:  $\mathsf{T}$  is a torsion class in  $\mathsf{M}(A)$  and  $\mathsf{S}^{\text{op}}$  is a torsion class in  $\mathsf{M}(B^{\text{op}})$ . As recalled above, these torsion classes extend to bimodule torsion classes  $\mathsf{T}, \mathsf{S}^{\text{op}} \subseteq \mathsf{M}(A \otimes B^{\text{op}})$ .

There is another way to view the bimodule torsion classes. Consider the two-sided ideals  $\mathfrak{a} \otimes B^{\text{op}}$  and  $A \otimes \mathfrak{b}^{\text{op}}$  in the ring  $A \otimes B^{\text{op}}$ . Then, as torsion classes in  $\mathsf{M}(A \otimes B^{\text{op}})$ , and with the notation of Definition 3.3, we have  $\mathsf{T} = \mathsf{T}_{\mathfrak{a} \otimes B^{\text{op}}}$  and  $\mathsf{S}^{\text{op}} = \mathsf{T}_{A \otimes \mathfrak{b}^{\text{op}}}$ .

In the next definition and theorem we have  $B = A$ , and only one torsion class  $\mathsf{T} \subseteq \mathsf{M}(A)$ . The properties of  $\mathsf{T}$  being quasi-compact, weakly stable and finite dimensional were introduced in Definition 3.4.

**Definition 8.2.** Let  $A$  be a flat central  $\mathbb{K}$ -ring, and let  $\mathsf{T}$  be a torsion class in  $\mathsf{M}(A)$ .

- (1) Let  $(\mathsf{R}\Gamma_{\mathsf{T}}, \sigma)$  be the copointed triangulated functor on  $\mathsf{D}(A)$  from Section 3.
- (2) Let  $P := \mathsf{R}\Gamma_{\mathsf{T}}(A) \in \mathsf{D}(A^{\text{en}})$ , with the morphism  $\rho : P \rightarrow A$  from Definition 7.11. So the pair  $(P, \rho)$  is a copointed object in  $\mathsf{D}(A^{\text{en}})$ .



(3) Let

$$G_{\mathsf{T}} : \mathsf{D}(A) \rightarrow \mathsf{D}(A)$$

be the triangulated functor  $G_{\mathsf{T}} := \mathsf{RHom}_A(P, -)$ . There is a morphism of triangulated functors  $\tau : \mathrm{Id} \rightarrow G_{\mathsf{T}}$  as in Definition 6.4, so the pair  $(G_{\mathsf{T}}, \tau)$  is a pointed triangulated functor on  $\mathsf{D}(A)$ .

(4) Let  $\mathsf{D}(A)_{\mathsf{T}\text{-tor}} \subseteq \mathsf{D}(A)$  and  $\mathsf{D}(A)_{\mathsf{T}\text{-com}} \subseteq \mathsf{D}(A)$  be the essential images of the functors  $\mathsf{R}\Gamma_{\mathsf{T}}$  and  $G_{\mathsf{T}}$  respectively.

**Theorem 8.3** (Noncommutative MGM Equivalence). *Let  $A$  be a flat central  $\mathbb{K}$ -ring, and let  $\mathsf{T}$  be a quasi-compact, weakly stable, finite dimensional torsion class in  $\mathsf{M}(A)$ . The following hold:*

- (1) *The functor  $G_{\mathsf{T}}$  is right adjoint to  $\mathsf{R}\Gamma_{\mathsf{T}}$ .*
- (2) *The copointed triangulated functor  $(\mathsf{R}\Gamma_{\mathsf{T}}, \sigma)$  and the pointed triangulated functor  $(G_{\mathsf{T}}, \tau)$  on  $\mathsf{D}(A)$  are idempotent.*
- (3) *The subcategories  $\mathsf{D}(A)_{\mathsf{T}\text{-tor}}$  and  $\mathsf{D}(A)_{\mathsf{T}\text{-com}}$  are full triangulated subcategories of  $\mathsf{D}(A)$ .*
- (4) *The functor*

$$\mathsf{R}\Gamma_{\mathsf{T}} : \mathsf{D}(A)_{\mathsf{T}\text{-com}} \rightarrow \mathsf{D}(A)_{\mathsf{T}\text{-tor}}$$

*is an equivalence of triangulated categories, with quasi-inverse  $G_{\mathsf{T}}$ .*

*Proof.* Define the triangulated functor  $F_{\mathsf{T}} : \mathsf{D}(A) \rightarrow \mathsf{D}(A)$  to be  $F_{\mathsf{T}} := P \otimes_A^{\mathsf{L}} -$ . According to Theorem 7.12, with  $B = \mathbb{K}$ , there is an isomorphism of triangulated functors  $F_{\mathsf{T}} \cong \mathsf{R}\Gamma_{\mathsf{T}}$ . On the other hand, according to Theorem 7.18, the pointed object  $(P, \rho)$  in  $\mathsf{D}(A^{\mathrm{en}})$  is idempotent. By Theorem 6.17(2), again with  $B = \mathbb{K}$ , there are equalities  $\mathsf{D}(A)_{\mathsf{T}\text{-tor}} = \mathsf{D}(A)_{F_{\mathsf{T}}}$  and  $\mathsf{D}(A)_{\mathsf{T}\text{-com}} = \mathsf{D}(A)_{G_{\mathsf{T}}}$  of full subcategories of  $\mathsf{D}(A)$ ; see Definition 6.6. Now the full subcategories  $\mathsf{D}(A)_{F_{\mathsf{T}}}$  and  $\mathsf{D}(A)_{G_{\mathsf{T}}}$  are triangulated, so item (3) is proved. Items (1), (2) and (4) here are items (1), (2) and (4), respectively, in Theorem 6.17.  $\square$

Here are two questions related to this theorem.

**Question 8.4.** Under which conditions is there an additive functor  $\Lambda : \mathsf{M}(A) \rightarrow \mathsf{M}(A)$ , such that the right adjoint to  $\mathsf{R}\Gamma_{\mathsf{T}}$  is  $G_{\mathsf{T}} = \Lambda$ ? Compare to the commutative situation in Theorem 0.1, where  $\Lambda = \Lambda_{\mathfrak{a}}$  is the  $\mathfrak{a}$ -adic completion functor. On the other hand, there are counterexamples where this fails (and these will be presented in the follow-up paper [Vs2]).

**Question 8.5.** By assumption the functor  $\mathsf{R}\Gamma_{\mathsf{T}}$  has finite cohomological dimension. In the commutative case (see Theorem 0.1), where  $\mathsf{R}\Gamma_{\mathsf{T}} = \mathsf{R}\Gamma_{\mathfrak{a}}$ , its right adjoint  $G_{\mathsf{T}} = \Lambda_{\mathfrak{a}}$  also has finite cohomological dimension. Is this true in the noncommutative case?

**Example 8.6.** Consider the ring  $A = \mathbb{Z}$ , the multiplicatively closed set  $S = \mathbb{Z} - \{0\}$ , and the torsion class  $\mathsf{T}_S$  in  $\mathsf{M}(\mathbb{Z})$ , as in Example 3.13. It can be shown that in this case the functor  $G_{\mathsf{T}}$  is the left derived functor  $\mathsf{L}\Lambda$ , where  $\Lambda : \mathsf{M}(\mathbb{Z}) \rightarrow \mathsf{M}(\mathbb{Z})$  is the “profinite completion” functor

$$\Lambda(M) := \varprojlim_k (M/k \cdot M).$$

Here  $k$  runs through the set of positive integers, with its partial order by divisibility.

**Example 8.7.** Suppose  $a$  is a regular normalizing element of the ring  $A$ . Recall that this means that  $A \cdot a = a \cdot A$ , and  $a$  is a non-zero-divisor. (It follows that there is an automorphism  $\gamma$  of the ring  $A$  such that  $a \cdot b = \gamma(b) \cdot a$  for all  $b \in A$ .) Let  $\mathfrak{a} \subseteq A$  be the ideal generated by  $a$ .

As already mentioned in Example 3.14, the torsion class  $\mathsf{T}_{\mathfrak{a}} \subseteq \mathsf{M}(A)$  is weakly stable. It can be shown (and will appear in the future paper [Vs2]) that  $\mathsf{T}_{\mathfrak{a}}$  is also quasi-compact and finite dimensional. Therefore Theorem 8.3 applies. Moreover, in this case the functor  $G_{\mathsf{T}}$  is nothing but  $\mathsf{L}\Lambda_{\mathfrak{a}}$ , the derived  $\mathfrak{a}$ -adic completion functor.

In the next definition and theorem the rings  $A$  and  $B$  can be distinct.

**Definition 8.8.** Let  $A$  and  $B$  be flat central  $\mathbb{K}$ -rings, and let  $\mathsf{T} \subseteq \mathsf{M}(A)$  and  $\mathsf{S}^{\mathrm{op}} \subseteq \mathsf{M}(B^{\mathrm{op}})$  be torsion classes. A complex  $M \in \mathsf{D}(A \otimes B^{\mathrm{op}})$  is said to have *symmetric derived  $\mathsf{T}$ - $\mathsf{S}^{\mathrm{op}}$ -torsion* if it satisfies these two conditions:

- (i) For every  $q$  the bimodule  $\mathrm{H}^q(\mathrm{R}\Gamma_{\mathsf{T}}(M))$  belongs to  $\mathsf{S}^{\mathrm{op}}$ .
- (ii) For every  $q$  the bimodule  $\mathrm{H}^q(\mathrm{R}\Gamma_{\mathsf{S}^{\mathrm{op}}}(M))$  belongs to  $\mathsf{T}$ .

**Theorem 8.9** (Symmetric Derived Torsion). *Let  $A$  and  $B$  be flat central  $\mathbb{K}$ -rings, and let  $\mathsf{T} \subseteq \mathsf{M}(A)$  and  $\mathsf{S}^{\mathrm{op}} \subseteq \mathsf{M}(B^{\mathrm{op}})$  be quasi-compact, weakly stable, finite dimensional torsion classes. Let  $M \in \mathsf{D}(A \otimes B^{\mathrm{op}})$  be a complex with symmetric derived  $\mathsf{T}$ - $\mathsf{S}^{\mathrm{op}}$ -torsion. Then there is an isomorphism*

$$\mathrm{R}\Gamma_{\mathsf{T}}(M) \cong \mathrm{R}\Gamma_{\mathsf{S}^{\mathrm{op}}}(M)$$

in  $\mathsf{D}(A \otimes B^{\mathrm{op}})$ . Moreover, this isomorphism is functorial in such complexes  $M$ .

*Proof.* Let  $P := \mathrm{R}\Gamma_{\mathsf{T}}(A) \in \mathsf{D}(A^{\mathrm{en}})$ , as in Definition 7.11. By Theorem 7.12 we know that

$$(8.10) \quad \mathrm{R}\Gamma_{\mathsf{T}} \cong P \otimes_A^{\mathrm{L}} -$$

as triangulated functors from  $\mathsf{D}(A \otimes B^{\mathrm{op}})$  to itself.

Recalling that  $(B^{\mathrm{op}})^{\mathrm{op}} = B$ , we see that there is a triangulated functor

$$\mathrm{R}\Gamma_{\mathsf{S}^{\mathrm{op}}} : \mathsf{D}(B^{\mathrm{en}}) \rightarrow \mathsf{D}(B^{\mathrm{en}}).$$

Applying this functor to the object  $B \in \mathsf{D}(B^{\mathrm{en}})$ , we obtain an object

$$Q := \mathrm{R}\Gamma_{\mathsf{S}^{\mathrm{op}}}(B) \in \mathsf{D}(B^{\mathrm{en}}).$$

After some possibly disorienting switches between rings and their opposites, and between  $A$  and  $B$ , we realize that Theorem 7.12 implies that

$$(8.11) \quad \mathrm{R}\Gamma_{\mathsf{S}^{\mathrm{op}}} \cong - \otimes_B^{\mathrm{L}} Q$$

as triangulated functors from  $\mathsf{D}(A \otimes B^{\mathrm{op}})$  to itself.

Because the derived tensor product is associative (see Proposition 5.17), we deduce from formulas (8.10) and (8.11) that there are isomorphisms

$$(8.12) \quad \mathrm{R}\Gamma_{\mathsf{S}^{\mathrm{op}}} \circ \mathrm{R}\Gamma_{\mathsf{T}} \cong (P \otimes_A^{\mathrm{L}} -) \otimes_B^{\mathrm{L}} Q \cong P \otimes_A^{\mathrm{L}} (- \otimes_B^{\mathrm{L}} Q) \cong \mathrm{R}\Gamma_{\mathsf{T}} \circ \mathrm{R}\Gamma_{\mathsf{S}^{\mathrm{op}}}$$

of triangulated functors from  $\mathsf{D}(A \otimes B^{\mathrm{op}})$  to itself.

Now let us write

$$N := \mathrm{R}\Gamma_{\mathsf{S}^{\mathrm{op}}}(M) \in \mathsf{D}(A \otimes B^{\mathrm{op}}).$$

Consider the morphism

$$\sigma_N^{\mathrm{R}} : \mathrm{R}\Gamma_{\mathsf{T}}(N) \rightarrow N$$

in  $\mathsf{D}(A \otimes B^{\mathrm{op}})$ . We claim this is an isomorphism. To prove this claim, we might as well forget about the right  $B$ -module structures, as we did in the proof of Theorem

7.12, relying on Lemma 7.9. So we can assume that  $N \in \mathbf{D}(A)$ . We are given that the cohomologies  $H^q(N)$  are in  $\mathbf{T}$ . By Proposition 3.7 the morphism  $\sigma_N^{\mathbf{R}}$  is an isomorphism. The conclusion of this paragraph is that there is a functorial isomorphism

$$(8.13) \quad \mathrm{R}\Gamma_{\mathbf{T}}(\mathrm{R}\Gamma_{\mathbf{S}^{\mathrm{op}}}(M)) \cong \mathrm{R}\Gamma_{\mathbf{S}^{\mathrm{op}}}(M)$$

for complexes of bimodules  $M$  satisfying condition (ii).

The same arguments, applied on the reverse side, lead us to conclude that there is a functorial isomorphism

$$(8.14) \quad \mathrm{R}\Gamma_{\mathbf{S}^{\mathrm{op}}}(\mathrm{R}\Gamma_{\mathbf{T}}(M)) \cong \mathrm{R}\Gamma_{\mathbf{T}}(M)$$

for complexes of bimodules  $M$  satisfying condition (i).

The combination of formulas (8.12), (8.13) and (8.14) gives the desired isomorphism.  $\square$

**Remark 8.15.** We can avoid the assumption that the rings  $A$  and  $B$  are flat over the base ring  $\mathbb{K}$ . This is done by taking K-flat DG ring resolutions of  $A$  and  $B$ , as in Remark 5.22. However, the technical complications of such a generalization are quite substantial. Therefore we have decided to restrict attention in the present paper to the flat case. The nonflat generalization will appear in the future paper [Vs1].

**Remark 8.16.** As was discovered by the first author of the present paper, the proof of [YZ, Theorem 1.23] was erroneous unless the base ring  $\mathbb{K}$  is a field. (Even then the proof of [YZ, Lemma 1.24] was incorrect; but that can be easily fixed when  $\mathbb{K}$  is a field.) For the purposes of the paper [YZ] this error was negligible, because for the remainder of that paper it was assumed anyhow that the base ring is a field.

Finding a correct proof of [YZ, Theorem 1.23] was one of our goals for some time. In Theorem 8.9 above we have accomplished that – almost. The caveat is that here we need to assume that the torsion classes  $\mathbf{T}$  and  $\mathbf{S}^{\mathrm{op}}$  are finite dimensional, and this condition did not appear in [YZ]. As for the other conditions: in [YZ] the torsion classes were assumed to be stable, and here only weak stability is needed. The condition “locally finitely resolved” in [YZ] implies quasi-compactness here.

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